Magnetoacoustic Waves and Instabilities in a Hall-Effect-Dominated Plasma

S. Palmgren

This report is intended for publication in a periodical. References may not be published prior to such publication without the consent of the author.

AKTIEBOLAGET ATOMENERGI
STUDSVIK, NYKÖPING, SWEDEN 1970
MAGNETOACOUSTIC WAVES AND INSTABILITIES IN A HALL-EFFECT-DOMINATED PLASMA

S. Palmgren

ABSTRACT

The dispersion equation is studied for small-amplitude plane harmonic waves in a compressible plasma moving perpendicular to a magnetic field with a constant fractional ionization. The modes of propagation are analysed mainly from a qualitative point of view and one of them is shown to be unstable due to the Hall effect. This mode has been previously analysed by other authors in connexion with MHD power generators but in a more restricted and isolated sense. The present work not only generalizes and modifies their results on this special mode, but also makes it possible to picture the whole spectrum of propagation modes in a simple and physically intelligible way.

Printed and distributed in May 1970.
LIST OF CONTENTS

1. INTRODUCTION .................................................. 3
2. BASIC ASSUMPTION AND EQUATIONS .................... 4
3. PROPAGATION MODES ........................................... 6
   3.1 Velikhov’s magnetoacoustic instability ............. 6
   3.2 Alfvén’s magnetoacoustic waves ................... 7
   3.3 Generalized magnetoacoustic wave ................. 7
      3.3.1 $k // v'_e$ ........................................ 8
      3.3.2 $k \perp v'_e$ ....................................... 9
4. NUMERICAL EXAMPLE AND DISCUSSION .................. 10
5. SUMMARY ......................................................... 11

ACKNOWLEDGEMENT

APPENDIX

REFERENCES

FIGURES
1. INTRODUCTION

Hydromagnetic waves in compressible media, magnetoacoustic waves (MAW), have been treated by Alfvén and Fälthammar (1963). They studied the propagation of small-amplitude, plane harmonic waves in the low-frequency approximation and introduced limitations to infinite conductivity and to a plasma at rest in a laboratory frame of reference. We shall omit these two latter restrictions and consider a finite conducting plasma moving perpendicular to a magnetic field and between two electrode plates, connected via an external load. The situation now is formally that of a magnetohydrodynamic power generator duct and we shall have this application in mind later on. In fact one branch of the MAW in compressible moving media has earlier been partly investigated in connexion with MHD power generation, originally by Velikhov (1962). This mode has been called the magnetoacoustic instability mode (MAI), because it represents a modified sound wave and turns out to be unstable provided that the electron drift velocity measured in the gas frame exceeds the speed of sound. This macroinstability is explained in terms of a coupling between the plasma parameters and the thermodynamic state of the gas, see McCune (1964). Inspection shows immediately that their derivation cannot be used to study the transition between MAI and MAW when the plasma velocity decreases to zero. The reason is simply that the MAI is derived for \( \mu \lambda \) small in some respect, formally that \( \mu \rightarrow 0 \), but the upper limit of \( \mu \lambda \) has not been specified in the above-mentioned papers. \( \mu \) is the permeability of free space and \( \lambda \) is the wavelength.

The main purpose of this note is now to synthetize Alfvén’s magnetoacoustic waves and the magnetoacoustic instability derived by Velikhov and consequently these will be called the generalized magnetoacous-
tic waves. For such a relatively complex treatment to be meaningful a sufficient condition is that we have pronounced hydromagnetic waves, which means that the Lundqvist number, i.e. the magnetic Reynolds number based on the Alfvén velocity $v_A$, or $L = \frac{\mu_0 \sigma t v_A}{\ell}$, should exceed unity. $\sigma$ is the conductivity and $\ell$ is a characteristic length. However, this criterion cannot be necessary when considering the whole spectrum of different modes, which also includes skin effects, because Lundqvist's criterion is a priori applicable to the modes of modified sound waves.

2. BASIC ASSUMPTION AND EQUATIONS

We consider a weakly or partially ionized compressible plasma moving perpendicular to a constant and homogeneous magnetic field $\mathbf{B}_0$ with a constant and homogeneous velocity $\mathbf{v}$. The moment equations for this three-component fluid are

$$\frac{\partial \rho}{\partial t} + \text{div} \rho \mathbf{v} = 0 \quad (1)$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \text{grad} \rho = j \times \mathbf{B} \quad (2)$$

and the adiabatic expansion law

$$\frac{dP}{dp} = \gamma \frac{P}{\rho} \quad (3)$$

where $\rho$ is the mass density and $P$ is the total ion and gas pressure.

The Ohm-Hall law, neglecting ion slip, reads

$$j + \mu_e j \times \mathbf{B} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (4)$$

where $\mu_e = e/m_e v_e$ is the electron mobility and $\sigma = \epsilon n_e \mu_e$ is the electron conductivity. $v_e$ is the effective collision frequency for momentum transfer. Finally, Maxwell's equations within the low-frequency approximation read
Now we immerse two electrode plates into the plasma parallel to the magnetic field and connected via an external load. Then the electric field strength in the unperturbed state is uniquely defined. Next we introduce two main assumptions: that of small magnetic Reynolds number, \( R_m = \mu \sigma v \ll 1 \), which means that the induced magnetic field \( B_i \) can be disregarded in the equations above when compared to the imposed field \( B_o \), and furthermore that of constant fractional ionization, or "frozen ionization" (see, for example, Witalis, 1968). This means that the mobility is inversely proportional to the gas density \( \rho \) but that the conductivity is independent of the state of the gas.

From these equations and assumptions we can derive the dispersion relation for small-amplitude, plane harmonic wave perturbations

\[ f = f_0 \exp \left\{ i (\omega t - k \cdot x) \right\} \]

and we restrict ourselves to wave vectors \( k \) perpendicular to the imposed magnetic field. We consider real wave vectors \( k \), and so the frequency \( \omega \) may in general be complex. This arbitrary choice is of no physical importance. After some lengthy algebraic manipulations in the usual fashion we arrive at the result

\[
\frac{a^2 - ik^2 g \cdot k}{w^2 - c^2} = 1 - \frac{iV}{w^2 - v^2_{ek}}
\]

where \( a \) now is the Alfvén velocity and \( c \) is the speed of sound. For practical purposes we have defined a complex phase velocity \( w = \omega/k = u + iv \) and 'prime' denotes a translational transformation to the gas frame of reference, \( \omega' = \omega - k \cdot v \), i.e. the Doppler effect. Further notations are \( V = k/\mu \sigma \), which is the penetration velocity (cf. skin effect),
and $v_{ek}'$, which is the projection on the wave vector of the electron drift velocity measured in the gas frame. Finally, $\mathbf{g} = \mathbf{j} \times \mathbf{B} / \rho$ is the Lorentz force per unit mass. This equation, which describes what we might call the generalized magnetoacoustic waves, is, apart from degenerated cases, of third degree in $w'$ and will then have three different modes of propagation. Later on we shall examine the shape, asymptotic behaviour and stability character of the different modes in the complex $w'$-plane for different values of the electron drift velocity, but first we shall rearrange equation (7) in such a way that comparisons with the special cases of MAI and MAW are possible.

3. PROPAGATION MODES

3.1 Velikhov's magnetoacoustic instability

For comparisons with the MAI we rearrange equation (7) to

$$w'^2 - c^2 - i\nu k^{-1} (w' - v_{ek}') =$$

$$z - \nu^{-1}(w' - v_{ek}')(w'^2 - c^2 + i k^{-2} \mathbf{g} \cdot \mathbf{k})$$

(8)

where $\nu^{-1} = \sigma / \omega B^2$ is the characteristic time for vorticity suppression (see, for example, Shercliff, 1965). For the RHS to be negligible in comparison with the terms in the LHS, we require formally that $\mu \rightarrow 0$, which results in a quadratic equation in $w'$ and has been derived also by, e.g., Rosa (1968, chapter 4). The two branches, which are always numbered consecutively from left to right, are sketched in figure 1 (a) and (b) for two values of the parameter $v_{ek}'$. The scales of the abscissa and ordinate are the same. The arrows on the different modes indicate direction of increasing wave vector and small circles denote the limit points, $k = \infty$ or 0. Due to the low-frequency approximation and the length scale limitation we have to exclude a neigh-
bourhood of these limit points symbolized by these circles. The first mode represents vorticity suppression and the second a modified sound wave which turns out to be unstable if the electron drift velocity measured in the gas frame exceeds the speed of sound. In figure 1 (b) we have inserted the unstable mode pictured in a \((u', w'_1)\)-plane, and the limiting value of the growth time for short wavelength is

\[
\tau_0 = \nu_0^{-1} = 2/\nu \left( \frac{v_{ek}}{c} - 1 \right)
\]

which is inversely proportional to the magnetic field strength squared.

### 3.2 Alfvén's magnetoacoustic waves

On the other hand, for comparisons with the MAW we write equation (7) in the form

\[
(w'^2 - c^2) (w' - v'_{ek} - iV) = (w' - v'_{ek}) (a^2 - ik^2 \mathbf{g} \cdot \mathbf{k})
\]

which in the case of a plasma at rest reduces to

\[
(w^2 - c^2) (w - iV) = wa^2
\]

and has been derived by Alfvén and Fälthammar (1963, chapter 3). Three modes should then appear, but for symmetry reasons only two of them are essential. The first and third ones, \(w_1\) and \(w_3\), represent forward and backward propagation of sound waves, and the last one, \(w_2\), is purely imaginary and pictures the skin effect. They are all stable and outlined in figure 2 (a). The limiting behaviour for short wavelengths is found to be

\[
\lim_{k \to \infty} \text{Im} (w_1, 3) = \nu \quad \text{and} \quad \lim_{k \to \infty} \text{Im} (w_2) \sim k^2/\mu \sigma
\]

### 3.3 Generalized magnetoacoustic wave

In studying the complete equation (7), we shall confine our attention to wave propagation either parallel or perpendicular to the elec-
tron drift velocity, or equivalently the electronic current. In both cases we shall outline the modes of propagation in the complex $w'$ plane for fundamentally different choices of the electron drift velocity.

3.3.1 $k // v_e'$

If $k$ is parallel to $v_e'$, equation (7) reduces to

$$\frac{a^2}{w'^2 - c^2} = 1 - i \frac{v}{w' - v_e'}$$

(12)

and we have actually three types of dispersion diagrams, $0 < v_e' < c$, $c < v_e' < c^*$ and $v_e' > c^*$, plus three or two degenerated cases, depending on whether we include $v_e' = 0$ as a degenerated case or not. Here $c^* = (c^2 + a^2)^{1/2}$ is the augmented sound velocity, numerically taken to be twice the ordinary sound velocity, see the discussion below of the numerical parameters. In figure 2 (a) we start with $v_e' = 0$, a plasma at rest, and this is the MAW as mentioned above. For increasing drift velocities, the main difference is at first only that the skin effect mode has a phase velocity essentially equal to the electron drift velocity. This is shown in figure 2 (b) where only the positive abscissa has been plotted, the backward modified sound remaining about the same as in figure 2 (a). At $v_e' = c$ the picture changes completely but, viewed as a bifurcation, still in a continuous way. First, at $v_e' = c$, figure 2 (c), we have a degenerated case where two of the branches partly cancel each other. In the range $c < v_e' < c^*$, figure 2 (d), the forward propagation of sound waves turns out to be unstable, and this is the MAI analogue as described by Velikhov. In fact they are found to be identical for short wave-lengths, and in the relevant interval of $v_e'$ this branch also has the same limit points, $c$ and $v_e'$. But for drift velocities above the augmented sound velocity, figure 2 (f), the limit
points are different from those obtained in figure 1. We may summarize the last picture 2 (f) as follows. The backward modified sound wave is unchanged except that the logarithmic decrement is lowered compared to a plasma at rest, the second mode constitutes an unstable modified sound wave (c and $c^*$ are limit points) propagating along the electron drift direction, and the third mode is the skin effect with a phase velocity still equal to the electron drift velocity.

At short wave-lengths the growth time of the unstable mode, denoted by subscript '2', is still of the same order as that derived by Velikhov, eq. (9). If higher order terms are included, obtained from eq. (12) by an iteration process, we get after series expansion in $(v_e' - c)/V$

$$\text{Im} \, w_2' = -\frac{\nu}{2k} \left( \frac{v_e'}{c} - 1 \right) \left[ 1 - \left( (v_e' - c)^2 \frac{V}{V} \right) \left( 1 + \left( 1 + \frac{1}{2} \left( \frac{a}{c} \right)^2 \right) \right) \right]$$

The expansion process itself is valid only if the square root is comparable to unity, which also means that the statements of eq. (9) are justified. Finally, in case $v'\gg c$ we get a growth time

$$\tau_1 = \frac{2c}{\nu v_e'} (1 - \alpha R_{me}^2)$$

where $R_{me} = \mu \sigma \nu_e'$ is the magnetic Reynolds number based on the electron drift velocity and $\alpha = \frac{1}{8\pi^2} \left( 1 + \frac{1}{2} \left( \frac{a}{c} \right)^2 \right)$. This means that the instability growth is somewhat decreased in comparison with Velikhov's results.

Finally, we consider waves perpendicular both to the imposed magnetic field and the electron drift in such a way that $(\mathbf{k}, \frac{v_e'}{c}, \mathbf{B})$ constitutes a right-handed cartesian coordinate system. Then
\[ k \cdot g = k (i \times B)/\rho = -e n_e \rho^{-1} k (v'_e \times B) = -a^2 k^2 v'_e/\kappa V \]

where \( \kappa = \mu_e B \) is the Hall coefficient. Then equation (7) reads

\[ \frac{a^2}{w''} (1 + i \frac{v'_e}{\kappa V}) = 1 - i \frac{V}{w'} \]

(15)

which is shown very schematically in figure 3 for not too high electron drift velocities. The first mode is a modified sound wave provided that we restrict ourselves to short wave-lengths. For increasing wave-lengths this mode is unstable and the critical wave-length is given by the intersection with the abscissa, \( V_{\text{crit}}^2 = v'_e c^*/\kappa \). This corresponds in general to wave-lengths which are very long when compared to a laboratory length scale. The asymptotic behaviour of the growth time when \( k \to 0 \)

\[ \lim_{k \to 0} \tau_2 = k a \sqrt{\frac{v'_e}{2\kappa V}} \sim O \left( k^{1/2} \right) \]

which shows that the instability growth has one or more extrema in the interval \( 0 < k < k_{\text{crit}} \). The physical interpretation and consequences of this long wave-length instability are still an open question. The second mode constitutes essentially the skin effect and the third represents, at least for short wave-lengths, a stable modified sound wave.

4. NUMERICAL EXAMPLE AND DISCUSSION

Concerning the application to MHD generator plasmas we shall consider a potassium-seeded argon plasma. For a detailed discussion of the plasma parameters, see for example Kerrebrock (1964) or Sakafo et al. (1969). First of all, from the gas parameters \( p = 1 \) atm \( T_a = 1500^0\text{K} \) and the adiabatic index \( \sigma = 1.668 \), and an imposed magnetic field strength \( B = 0.8 \) T, we obtain a sound velocity \( c = 720 \) m/s
and an Alfvén velocity \( v = 1270 \text{ m/s} \), so that the augmented sound velocity is about twice the ordinary sound velocity. From the plasma parameters \( n_e = 10^{19} \text{ m}^{-3} \), \( T_e = 2500 \text{ K} \) and \( m_e/m_a = 1.36 \cdot 10^{-5} \) we have an electron-neutral collision frequency \( v_{ea} = 2.3 \cdot 10^{10} \text{ s}^{-1} \) and for electron-electron collisions \( v_{ee} = 2.7 \cdot 10^8 \text{ s}^{-1} \), so that the plasma is partially ionized. Furthermore the conductivity \( \sigma = \frac{n_e e^2}{m_e v_{ea}} \approx 12 \text{ A/Vm} \). An obtainable value of the current is up to \( 2 \cdot 10^4 \text{ A/m} \), which means that the electron drift velocity is below \( 10^4 \text{ m/s} \) (the thermal velocity of the electrons is about \( 2 \cdot 10^5 \text{ m/s} \)). The characteristic time for vorticity suppression is \( \nu^{-1} = 4 \cdot 10^{-2} \text{ s} \) and the criterion for \( \mu \lambda \) to be small in eq. (8) (only concerning the unstable mode) is numerically \( \lambda^2 < 5 \cdot 10^2 \), i.e. wave-lengths smaller than about 7 metres.

We may note that, although this is larger than any generator size up to date, it may be important in other situations where, for example, the conductivity is increased. The growth time of the instability is here correctly expressed by eq. (9) \( \tau_o \approx 6 \text{ ms} \).

Finally, the instability perpendicular to \( \nu_e \) occurs if

\[
\lambda > \frac{2\pi}{\mu \sigma} \sqrt{\frac{\kappa}{\nu_e c}} = 400 \text{ m}
\]

and is of no interest in this context.

5. SUMMARY

The propagation of magnetoacoustic waves in a Hall effect plasma has been treated in the present paper. The evolution of the different modes with increasing plasma velocity was followed, mostly described in a qualitative way. It was found that the skin effect mode could with advantage be studied in the electron frame of reference, whereas hydromagnetic effects, including vorticity suppression, occur-
red in the gas frame. One of these modes, called MAI mode, has partly been analyzed by Velikhov and many others in a simplified way. The two descriptions of this instability, propagating preferentially in a direction parallel to the electron drift, are still the same with regard to the instability criterion and within the limit of small wave-lengths. A criterion for the applicability of their simplified treatment was derived and we found that \( R_{me} = \mu \sigma \lambda v_e \) should be small. We may first note that this criterion concerns only one special mode and not all of them, and further that, even if the magnetic Reynolds number \( R_m = \mu \sigma \lambda v \) and/or the Lundqvist number \( L_u = \mu \sigma \lambda v_A \) are small, this might not be true for \( R_{me} \) because of a strong Hall effect.

From the description of the generalized magnetoacoustic waves we have also found that a new instability might occur for long wave-lengths, \( k < k_{cr} = \mu \sigma \sqrt{v_e} c^*/k \), but due to the infinitesimal perturbation method used cautiousness is demanded on this point (for various aspects, see, for example, Grad, 1965). Anyhow this instability is of no interest in a laboratory scale, e.g. in a MHD generator plasma.
ACKNOWLEDGEMENT

The author is much indebted to Dr. E. Dahlberg for stimulating discussions and valuable criticism.

This work has been financially supported by the Swedish Board of Technical Development, contract 69-187/U115.
APPENDIX

As soon as \( v_e > c \), it is useful to write eq. (12) in the form

\[
\begin{align*}
w_2' &= c \left[ 1 + \left( \frac{a}{c} \right)^2 \frac{v_e' - w_2'}{v_e - w_2' + iV} \right]^{1/2}
\end{align*}
\]

where subscripts denote the modes under consideration. For short wavelengths, \( V \to \infty \), we obtain by an iteration process

\[
\begin{align*}
w_2'(0) &= c \\
w_2'(1) &= c \left[ 1 + \left( \frac{a}{c} \right)^2 \frac{v_e' - c}{v_e - c + iV} \right]^{1/2}
\end{align*}
\]

or in terms of \( p = (v_e' - c)/V \)

\[
\begin{align*}
w_2'(1) &= c \left[ \frac{\left( 1 + p^2 \left( \frac{c}{c} \right)^2 \right) - i \left( p^2 \left( \frac{a}{c} \right)^2 \right)^{1/2}}{1 + p^2} \right]
\end{align*}
\]

where superscript ( ) denotes the number of iteration steps. Concerning the imaginary part we obtain

\[
\text{Im } w_2'(1) = -c \left[ \frac{\left( p^2 \left( \frac{a}{c} \right)^4 + \left( 1 + p^2 \left( \frac{c}{c} \right)^2 \right) \right)^{1/2}}{2 \left( 1 + p^2 \right)} \right] 
\]

and by series expansion in \( p \), for small \( p \), we get

\[
\text{Im } w_2'(1) = -\frac{1}{2} \frac{a^2}{c} \left[ 1 - p^2 \left( 1 + \frac{1}{2} \left( \frac{a}{c} \right)^2 \right) \right]^{1/2}
\]

or

\[
\begin{align*}
w_{12} &= -\frac{\alpha}{2} \left( \frac{v_e' - c}{c} \right) \left[ 1 - \left( \frac{v_e' - c}{V} \right) \left( 1 + \frac{1}{2} \left( \frac{a}{c} \right)^2 \right) \right]^{1/2}
\end{align*}
\]

Higher order iterations are negligible if \( |w_2'(0) - w_2'(1)| \) is of the order \( p \) or smaller.
REFERENCES

1. ALFVÉN H. and FÄLTHAMMAR C.-G.,
Cosmical electrodynamics.

2. GRAD H.,
Variational principle for a guiding-center plasma.

3. KERREBROCK J. L.,
Nonequilibrium ionization due to electron heating. I. Theory.

4. McCUNE J. E.,
Wave growth an instability in partially-ionized gases.
Int. symp. on magnetohydrodynamic electrical power generation.

5. ROSA R. J.,
Magnetohydrodynamic energy conversion.

6. SAKAO F. and SATO H.,
Nonequilibrium electrical conductivity of a potassium-seeded argon plasma.

7. SHERCLIFF J. A.,
A textbook of magnetohydrodynamics.

8. VELIKHOV E. P.,
Hall instability of current-carrying slightly-ionized plasmas.

9. WITALIS E. A.,
Rotational motion of magnetized plasmas.
Fig. 1. - Velikhov's magnetoacoustic instability.

\[ w'^2 - c^2 = i \frac{\nu}{k} (\omega' - v'_{ek}) \]

(a) \( v'_{ek} < c \), (b) \( v'_{ek} > c \).
Fig. 2. - Generalized magnetoacoustic waves, $k \parallel v'_e$. 

\[
\frac{\alpha^2}{v'^2 - c^2} = 1 - i \frac{\nu}{v' - v'_e}
\]

(a) $v'_e = 0$, Alfvén waves. (b) $0 < v'_e < c$. (c) $v'_e = c$.
(d) $c < v'_e < c^*$. (e) $v'_e = c^*$. (f) $v'_e > c^*$.
Fig. 3. - Generalized magnetoacoustic waves, $k \perp v_o'$.

$$\frac{a^2}{w'^2 - c^2} \left(1 + i \frac{v_o'}{w'} \right) = 1 - i \frac{v}{w'}$$