Analysis of Linear MHD Power Generators

E. A. Witalis

AKTIEBOLAGET ATOMENERGI
STOCKHOLM, SWEDEN 1965
ANALYSIS OF LINEAR MHD POWER GENERATORS

E. A. Witalis

Abstract

The finite electrode size effects on the performance of an infinitely long MHD power generation duct are calculated by means of conformal mapping. The general conformal transformation is deduced and applied in a graphic way. The analysis includes variations in the segmentation degree, the Hall parameter of the gas and the electrode-insulator length ratio as well as the influence of the external circuitry and loading.

A general criterion for a minimum of the generator internal resistance is given. The same criterion gives the conditions for the occurrence of internal current leakage between adjacent electrodes. It is also shown that the highest power output at a prescribed efficiency is always obtained when the current is made to flow between exactly opposed electrodes.

Curves are presented showing the power-efficiency relations and other generator properties as depending on the segmentation degree and the Hall parameter in the cases of axial and transverse power extraction.

The implications of limiting the current to flow between a finite number of identical electrodes are introduced and combined with the condition for current flow between opposed electrodes. The characteristics of generators with one or a few external loads can then be determined completely and examples are given in a table. It is shown that the performance of such generators must not necessarily be inferior to that of segmented generators with many independent loads. However, the problems of channel end losses and off-design loading have not been taken into consideration.

Printed and distributed in February 1965
<table>
<thead>
<tr>
<th>LIST OF CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>3</td>
</tr>
<tr>
<td>1. Basic equations and assumptions</td>
<td>4</td>
</tr>
<tr>
<td>2. The conformal mapping of the electrode wall</td>
<td>8</td>
</tr>
<tr>
<td>3. The conformal mapping of the MHD channel</td>
<td>12</td>
</tr>
<tr>
<td>4. Electrode potentials</td>
<td>13</td>
</tr>
<tr>
<td>5. Electrode current</td>
<td>15</td>
</tr>
<tr>
<td>6. Resistance between internally connected electrodes</td>
<td>16</td>
</tr>
<tr>
<td>7. Power and efficiency</td>
<td>17</td>
</tr>
<tr>
<td>8. Best positions of segmented generator electrodes</td>
<td>18</td>
</tr>
<tr>
<td>9. Power, efficiency and internal resistance of the segmented generator with opposed electrodes</td>
<td>19</td>
</tr>
<tr>
<td>10. The single load generator</td>
<td>21</td>
</tr>
<tr>
<td>11. Maximum power output from a single load channel</td>
<td>23</td>
</tr>
<tr>
<td>12. Generators with a finite number of electrodes</td>
<td>24</td>
</tr>
<tr>
<td>13. Performance of generators with a finite number of electrodes</td>
<td>26</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>28</td>
</tr>
<tr>
<td>Table</td>
<td>30</td>
</tr>
<tr>
<td>Figures</td>
<td>31</td>
</tr>
</tbody>
</table>
Introduction

The performance of any method for MHD power generation from a moving gas will be determined mainly by the Hall effect, i.e. the drift motion of the charge carriers in the gas flow direction. As the extracted electrical power always originates from the braking of the gas flow, an efficient Lorentz force is a necessary condition, implying a strong magnetic field in the duct as well as good electrical conductivity of the gas. These two implications are exactly those which cause a strong Hall effect and, as explained in ref. 1, necessitate only two possible and basically different generator configurations, one suppressing the drift motion, the other taking advantage of it. A large number of theoretical investigations describe various properties of these two generator types, usually called the segmented generator and the Hall generator respectively. Practically all of these investigations have had to restrict their treatment to the two limiting cases when the drift motion is completely suppressed or is fully developed, both cases requiring an infinite number of electrodes. This is often a very crude approximation, e.g. Kerrebrock (2) has shown that the elimination of such a simplification will lead to a very strong deterioration in the performance of a MHD generator with nonequilibrium ionization.

Hurwitz, Kilb and Sutton (3) used conformal mapping to determine the current distribution and the internal resistance of a segmented MHD generator. They assumed the current distribution in the channel central portion to be uniform and perpendicular to the gas flow direction and they treated the case when the electrode length in the gas flow direction is equal to that of the insulator. Such a case will here be called that of $b = 1$. Deducing a more general conformal transformation, Witalis (4) extended their investigation to values of $b$ differing from unity and found the condition for a minimum of the generator internal resistance. The same transformation was used by Sutton (5) to obtain characteristics of a $b = 1$ segmented generator having exactly opposed electrodes. Dzung's analysis (6) is also
restricted to the $b = 1$ case but it is more general than previous treatments as it is not limited to any specific generator configuration.

The generator internal current distribution will here be considered as the solution of a potential problem which is solved by conformal mapping. The understanding of the electrode effects will be made easier as the physical problem as well as its representation in the complex plane will be studied with reference to their conformally transformed and simpler counterparts.

1. Basic equations and assumptions

Using rationalized MKS units the generalized Ohm's law is taken as (7)

$$j = \sigma (E + V \times B) + \omega_e \tau_e j x B/B + \omega_e \tau_e i \tau_e (j x B) x B/B^2 \quad (1)$$

implying that only steady conditions are considered and that the gradients of the charged components of the gas have been neglected. The first right hand side term is the current component given by the scalar electron conductivity $\sigma$ times the electric field $E_o$ in the gas frame which moves with the velocity $V$ relative to the channel

$$E_o = E + V \times B \quad (2)$$

and $B$ is the magnetic field strength. The second term of Eq. (1) gives rise to the Hall effect, $\omega_e$ being the electron cyclotron frequency

$$\omega_e = eB/m_e \quad (3)$$
e and \( m_e \) denoting the electron charge and mass. \( \tau_e \) is the average time between collisions randomizing the electron velocity. The last term on the right hand side introduces the effect of ion slip, \( \omega_i \) and \( \tau_i \) being the corresponding frequency and deflection time for the ions.

The Maxwell equations for the magnetic field are taken as

\[
\text{curl } \mathbf{B} = 0 \tag{4}
\]

\[
\text{div } \mathbf{B} = 0 \tag{5}
\]

Eq. (4) is generally a good approximation for MHD generators where the externally applied static field may be orders of magnitude larger than the induced magnetic effects.

As there is no time dependence, and as space charge quasi-neutrality is assumed, the Maxwell equations for the static electric field read

\[
\text{curl } \mathbf{E} = 0 \tag{6}
\]

\[
\text{div } \mathbf{E} = 0 \tag{7}
\]

Eq. (6) proves that a potential function can be defined as

\[
\mathbf{E} = -\text{grad } \varphi \tag{8}
\]

The magnetic field is assumed to be constant and it is taken in the positive z-direction of a cartesian coordinate system as shown in Fig. 1. The velocity \( V \) of the conducting gas is assumed to be constant and uniform across the channel implying that viscosity is neglected. The same type of flow would be obtained by considering an incompressible gas with an initially uniform velocity or a turbulent flow with a full velocity profile.
Fig. 1 shows the rectangular and constant cross section of the channel and a few electrodes. Their internal current connection as well as the external circuitry will be discussed later. The channel height is taken to be unity and the distance between the electrode walls is \( d \). The channel is assumed to be very long, i.e., effects associated with its entrance and exit are not considered. The conformal mapping allows the determination of the current distribution in a \( x-y \) plane only and constant conditions in the magnetic field direction are therefore assumed. Electrodes and insulators are considered to be perfect. The segmentation length, i.e., the combined distance in the gas flow direction of an electrode and an insulator, is \( 2p \), the insulator length being \( b\rho \), \( 0 < b < 2\rho \).

In the channel center the current flow is assumed to be uniform. It will be shown that a distance \( d/2 > 2p \) is necessary for the strongly non-uniform current flow near the electrode wall to become approximately homogeneous.

When there is no current in the \( z \)-direction the components of Eq. (1) can be written as

\[
\begin{align*}
  j_x & = \sigma_{\text{eff}} \left[ E_x + \beta_e (E_y - VB) \right] \\
  j_y & = \sigma_{\text{eff}} \left[ -\beta_e E_x + E_y - VB \right]
\end{align*}
\]

where \( \beta_e \) is the reduced Hall parameter for electrons taking ion slip into account

\[
\beta_e = \omega_e \tau_e (1 + \omega_e \tau_e \omega_i \tau_i)^{-1} = \tan \Theta
\]

and the effective electrical conductivity is given by

\[
\sigma_{\text{eff}} = \sigma_0 \beta_e \left[ \omega_e \tau_e (1 + \beta_e^2) \right]^{-1} = \sigma_0 (1 + \beta_e^2)^{-1}
\]
The above restrictions on \( V \) and \( B \) are such that the electric field in the moving gas frame, \( \mathbf{E}_0 \), is also irrotational and thus a potential \( \phi \) can be defined

\[
\mathbf{E}_0 = -\text{grad} \phi
\]  

(12)

The current components can now be expressed as

\[
j_x = J_{\text{eff}} \left( -\frac{\partial \phi}{\partial x} - \beta e \frac{\partial \phi}{\partial y} \right)
\]

\[
j_y = J_{\text{eff}} \left( \beta e \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} \right)
\]  

(13)

The relation between the potentials \( \phi \) and \( \phi \) is simply

\[
\phi = VBy + \varphi
\]  

(14)

Applying the Maxwell equation for conservation of charge

\[
\text{div} \mathbf{j} = 0
\]  

(15)

on Eq. (13) it is proved that the Laplace equation for the potential \( \phi \)

\[
\Delta \phi = 0
\]  

(16)

is satisfied. Hence a stream function \( \psi \) can be associated to the potential function \( \phi \) by means of the Cauchy-Riemann equations

\[
\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}
\]

\[
\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}
\]  

(17)
Introducing \( \psi \) in Eq. (13) the current components can now be found from a function \( Z \)

\[
\begin{align*}
j_x &= \frac{\partial Z}{\partial y} \\
\frac{\partial Z}{\partial x}
\end{align*}
\]  

where \( Z \) is given by

\[
Z = -\sigma_{\text{eff}}(\psi + \beta \phi)
\]  

Consider two \( x-y \) planes separated by unit distance. Let two points \( P_1 \) and \( P_2 \) on one plane be connected by a line \( P_1P_2 \). This line and its perpendicular projection on the other plane define two opposite edges, the two straight lines between the points and their projections forming the remaining edges of a surface. The current \( I \) crossing this surface is given by

\[
I = \int_{P_1}^{P_2} j_x \, dy - j_y \, dx = Z(P_2) - Z(P_1)
\]  

or, when considering the problem as two-dimensional, the current crossing a line is given by the difference in the values of \( Z \) for the line end points.

2. The conformal mapping of the electrode wall

The \( x \)-axis in Fig. 2 represents the electrode wall which has been transformed into a \( w \)-plane so that the heavily drawn lines correspond. It will be shown that potential and current distributions are easily found in the \( w \)-plane geometry because such a transformation of the \( z \)-plane electrode wall will make the homo-
geneous channel center conditions valid near the slanting w-plane electrodes and insulators.

The appropriate Schwarz-Christoffel transformation will be deduced. It transforms \( \text{Im} \ z > 0 \) into the area above the broken line of the w-plane so that the points \( n \phi, \ n \) even, of the real axes correspond and \( z = a \phi \) transforms into \( w = A \), \( z = b \phi \) into \( w = B \) etc. Using any standard treatise on complex analysis the transformation derivative can be written down directly as an infinite product

\[
\frac{dw}{dz} = k \cdots (z + 2\rho - a \rho)(z + 2\rho - b \rho)^{0/\pi - 1/2} (z)^{-0/\pi - 1/2} .
\]

\[
(z - a \rho)(z - b \rho)^{0/\pi - 1/2} .
\]

which can be compressed by applying the product expansion of the sine function

\[
\frac{dw}{dz} = k \left[ \sin \frac{\pi}{2} \left( \frac{z}{\phi} - b \right) \right]^{0/\pi - 1/2} \left( z + \frac{\pi b}{2 \rho} \right)^{-0/\pi - 1/2} \left( z - \frac{\pi a}{2 \rho} \right) .
\]

When the constant \( k \) is given the value

\[
k = e^{-i\alpha}
\]

it is readily found that the real axis of the z-plane transforms with the correct angles and directions into the w-plane.

The homogeneous channel center conditions are identical for the two planes, implying that the transformation derivative must have the limit unity with increasing values of \( y \). It is then possible to find a relation which expresses the constant \( a \) in terms of the scaled insulator length \( b \) and the angles \( \theta \) and \( \alpha \)

\[
a = 2\alpha/\pi + b(1/2 - \theta/\pi)
\]
The transformation as shown in Fig. 2 is obtained only when the condition $a \leq b$ is satisfied. Ref. 4 proves that the case $a > b$ leads to current flow between adjacent electrodes. Such a kind of current leakage is unfavourable and it will not be considered here. Eq. (23) then gives a lower limit for the insulator length $b_\rho$, its upper limit being given by the segmentation length $2\rho$.

$$2\alpha/(\pi/2 + 0) \leq b < 2$$  \hspace{1cm} (24)

Insertion of $z = iy$ in Eq. (21a) shows that the variation of the transformation derivative is determined by the function $\tanh \frac{\pi y}{2\rho}$ and that homogeneity can be considered established at a distance equal or larger than $2\rho$ from the electrode wall.

In the homogeneous zone the transformation simplifies to a pure translation

$$w = z + \rho \Delta_u + i\rho \Delta_v$$  \hspace{1cm} (25)

where the shifts $\Delta_u$ and $\Delta_v$ are given by

$$\left[ \int_0^\infty \frac{dw}{dz} - 1 \right] dz \bigg|_{z=iy} = \rho \Delta_u + i\rho \Delta_v$$  \hspace{1cm} (26)

$\Delta_u$ and $\Delta_v$ are considered to be functions of the three variables $b$, $\theta$ and $\alpha$, the quantity $a$ being eliminated in the integral of Eq. (26) by the use of Eq. (23). The angle $\alpha$ will be shown to be determined by the loading of the generator or the external circuitry. It is desirable to be able to distinguish the influence of $\alpha$ on the shifts from that of $b$ and $\theta$ which represent physical properties of the generator duct. An elementary although somewhat lengthy calculation gives the result
\[ \pi \Delta_u = I_1 \sin \beta - I_2 \sin \gamma + \pi (1 - b)/2 \]

\[ \pi \Delta_v = I_1 \cos \beta - I_2 \cos \gamma - I_3 - \left[ \ln(2 - 2 \cos \beta) \right] / 2 \]  

where

\[ \beta = 2 \alpha - \vartheta_b - \pi b/2 , \quad \gamma = 2 \alpha - \vartheta_b + \pi b/2 \]

and the integrals do not depend on \( \alpha \)

\[ I_1 = \int_0^1 t^{1/2 - 0/\pi} \left( 1 - 2 \cos \pi b + t^2 \right)^{-1} dt \]

\[ I_2 = \int_0^1 t^{-1/2 - 0/\pi} \left( 1 - 2 \cos \pi b + t^2 \right)^{-1} dt \]

\[ I_3 = \int_0^1 (1 - t^{-1/2 - 0/\pi})(1 - t)^{-1} dt \]

The integral \( I_3 \) can be expressed in terms of the \( \psi \)-function defined as the derivative of the logarithm of the \( \Gamma \)-function. It has not been found possible to express \( I_1 \) and \( I_2 \) in known mathematical functions except for certain values of \( b \).

The shift \( \Delta_v \) is generally positive but \( \Delta_u \) is a negative quantity when power is generated in the channel. When considered as functions of \( b \), both \( \Delta_v \) and the absolute value of \( \Delta_u \) attain minima for

\[ b = 2 \alpha/(\pi/2 + 0) \]  

(28)
3. The conformal mapping of the MHD channel

The z-plane of Fig. 3 shows how the channel can be considered as constructed from two infinitely long half-strips of complex planes having the center line \( y = d/2 \) in common. As \( d/2 \geq 2\rho \) homogeneity is established there and no discontinuity occurs for currents and potentials which are assumed to be identical to those of the corresponding central region of the w-plane. For this plane it is clear that the directions of field lines, currents and equipotentials are the same everywhere between the slanting electrodes and insulators.

A comparison between the w- and the z-plane gives the geometrical interpretation of the lengths \( -\rho \Delta_u \) and \( \rho \Delta_v \). They are the components of that shift which the point \( z = 0 \) would experience in a conformal transformation \( z_1 = w_1 \) with the origin \( w_1 = z_1 = 0 \) situated at a fixed point on the common center line.

With no loss of generality we can assign the potentials \( \phi = 0 \) and \( \phi = 0 \) to that electrode having its right hand side end coinciding with the origins of the complex planes. The complete potential distribution will then be determined if the basic assumptions and simplifications given above are supplemented by a condition governing current or potentials at the central portion of the duct.

Generally, the highest local power output density and efficiency are obtained when the \( j \times B \) force exactly opposes the gas flow. Then for a segmented MHD generator with separate loads and finite size electrodes it is natural to assume either the current flow in the channel center portion to be perpendicular to the channel center axis or externally connected electrodes to be geometrically opposed. In the first case the condition of total internal current connection requires a non-symmetrical position of two connected electrodes, i.e. one electrode row has to be shifted a distance \( -\rho(2\Delta_u + b) \) in the gas flow direction relative to the opposite row. The calculations
given in ref. 4 prove that this shift is always less than one segmentation length $2\rho$. In the second case the same condition implies that the current in the central part of the channel has a component which is not perpendicular to the gas flow.

In the following there will not be an explicit restriction on the current direction in the central part of the channel. Instead the potential $\varphi = 0$ will be prescribed at a point there. The physical interpretation of this is simply that the relative position of two externally short-circuited electrodes can be treated as a known constant or as a variable. This method when investigating various generator configurations was introduced by ref. 6.

4. Electrode potentials

The top row electrode with the arbitrary position $(x_o; d)$ is assumed to be short-circuited to the electrode at the origin which has the potential $\varphi = 0$. Because of symmetry the electrostatic potential at the point $(x_2; d/2) = (x_o/2 - \rho(2 - b)/2; d/2)$ will also be zero. Eq. (14) then gives the potential $\varphi = \varphi_o$ at $(x_2; d/2)$

$$\varphi_o = VBd/2$$  \hspace{1cm} (29)

Fig. 3 shows that the equipotentials $\varphi = \text{const.}$ of the $w$-plane are parallel to the slanting electrodes. The stream lines $\psi = \text{const.}$ are in the direction of the electric field:

$$\varphi = A \text{Im}(w e^{i\omega})$$  \hspace{1cm} (30)

$$\psi = -A \text{Re}(w e^{i\omega})$$

The constant $A$ is recognized as the magnitude of the electric field strength in the moving gas frame of the channel center. Combining Eqs (25), (29) and (30), $A$ can be found
\[ A \text{Im} \left[ \left( x_2 + \rho \Delta_u \right) + i(d/2 + \rho \Delta_v) \right] e^{i\alpha} = VBd/2 \]
giving
\[ A = VBd \left[ x_0 \sin\alpha + d \cos\alpha + 2 \rho q \sin\alpha \right]^{-1} \tag{31} \]
where
\[ q = \Delta_v \cot\alpha + \Delta_u + b/2 - 1 \tag{32} \]

The potentials being prescribed, the current flow pattern is determined by Eq. (13) or more simply, by noting that the current flow lines \( \psi' = \text{const.} \) are parallel to the insulators of the \( w \)-plane:

\[ \psi' = \text{Im} \left[ w e^{i(\pi/2 + \alpha - \theta)} \right] \tag{33} \]

The current flow line \( \psi' = 0 \) is considered. The abscissa \( x_1 \) of its intersection with the channel center line \( y = d/2 \) is determined by the equation

\[ \text{Im} \left[ \left( x_1 + \rho \Delta_u \right) + i(d/2 + \rho \Delta_v) \right] e^{i(\pi/2 + \alpha - \theta)} = 0 \]
giving
\[ x_1 = -\rho \Delta_u - (d/2 + \rho \Delta_v) \tan(\theta - \alpha) \tag{34} \]

At the point \( (x_1 - \rho; d/2) \) the potential will be \( \phi_k/2 \), \( \phi_k \) denoting the potential of the top row electrode \( B_1B_2 \) which emits the current going to the electrode \( A_1O \) at the origin. From Eqs. (30) and (34) the potential \( \phi_k \) is found:

\[ \phi_k = A \left[ (d + 2 \rho \Delta_v) \cos\theta / \cos(\theta - \alpha) - 2 \rho \sin\alpha \right] \tag{35} \]
and the electrostatic electrode potential \( \varphi_k \) is then obtained from Eq. (14):

\[
\varphi_k = \Phi_k - \text{VBd} = -\text{VBd} \left[ x_o + d \tan(\theta - \alpha) + 2 \rho p \right].
\]

\[
\cdot \left[ x_o + d \cot \alpha + 2 \rho q \right]^{-1}
\]

(36)

where

\[
p = \Delta_v \tan(\theta - \alpha) + \Delta_u + b/2
\]

(37)

5. Electrode current

The total current \( I_e \) flowing to one electrode is obtained as the difference \( \Delta Z \) between the two values which the function \( Z \), Eq. (19), attains at two points situated at adjacent electrode current dividing lines. A simple way to calculate \( \Delta Z \) is found by considering the end points \( w(A_1) = 2 \rho e^{i(\pi - \alpha)} \cos(\theta - \alpha)/\cos \theta \) and \( w = 0 \) of the \( w \)-plane electrode having the potentials zero. Only the stream function \( \psi \) varies along an electrode. This fact and the equations (19), (20) and (30) yield the electrode current:

\[
I_e = \Delta Z = -\sigma_{eff} \Delta \psi = \sigma_{eff} A \text{Re} \left[ \{ 0 - w(A_1) \} e^{i\alpha} \right] = 2 \rho A \sigma_{eff} \cos(\theta - \alpha)/\cos \theta
\]

(38)

Applying the equations (10), (11) and (31), \( I_e \) can be written

\[
I_e = 2 \rho \text{VBd} \sigma_1 \left[ x_o + d \cot \alpha + 2 \rho q \right]^{-1} \left[ \tan \theta - \tan(\theta - \alpha) \right]^{-1}
\]

(38a)

and the equations (10) and (11) prove that the reduced conductivity
\( \sigma_1 \) is related to the scalar electronic conductivity \( \sigma_0 \) by

\[
\sigma_1 = \sigma_0 \left( 1 + \omega \tau \omega \tau \right)^{-1}
\]

(39)

6. Resistance between internally connected electrodes

The center line conditions are corresponding and identical for the \( w \)- and the \( z \)-plane. This fact will make electrode potentials and currents to be invariants of the conformal transformation and a simple geometrical interpretation of the finite size electrode effects on the internal resistance is possible. Consider the current strip between the electrodes \( OA_1 \) and \( B_1 B_2 \) in the \( w \)-plane. The current path length \( L \) is given by

\[
L = (d + 2\rho \Delta_y) / \cos(\theta - \alpha) - 2\rho \sin\alpha / \cos\theta
\]

(40)

Tonk's theorem (1) gives the current density

\[
j = \sigma_1 \phi_k / L
\]

(41)

which multiplied by the strip width \( 2\rho \cos(\theta - \alpha) \) yields the electrode current

\[
I_e = 2\rho \sigma_1 \phi_k \cos(\theta - \alpha) / L
\]

The internal resistance \( R_i \) of the current strip is related to \( \phi_k \) and \( L \) as

\[
R_i = \phi_k / I_e = L \left[ 2\rho \sigma_1 \cos(\theta - \alpha) \right]^{-1} = \left[ (d + 2\rho \Delta_y) \cos^2(\theta - \alpha) - 2\rho \sin\alpha \cos^{-1} \theta \cos^{-1}(\theta - \alpha) \right] / (2\rho \sigma_1)
\]

(42)
showing that only the shift $\Delta_v$ and not $\Delta_u$ affects the internal resistance. Therefore, in order to obtain the smallest possible internal resistance of any given generator configuration, the scaled insulator length $b$ should satisfy the Eq. (28) which yields a minimum of the shift $\Delta_v$ as well as minima of the current path length $L$ and the internal resistance $R_i$.

7. Power and efficiency

The generated power output $P$ from a volume $W$ is given by

$$P = \int_{W} -E \cdot \mathbf{j} \, dW \quad (43)$$

When Eqs. (8) and (15) are valid Eq. (43) can be written as an integral taken over the volume surface $S$

$$P = \int_{S} \mathbf{\varphi j} \cdot d\mathbf{S} \quad (44)$$

Consider the power output $P_e$ from a current strip that goes between two electrodes. Only the electrode surfaces will then contribute to the integral (44) which becomes

$$P_e = -\varphi_k l_e =
= 2\rho \tau \left(VBd\right)^2 \left[x_o + d \tan(\theta - \alpha) + 2\rho p \right] x_o + d \cot \alpha +
+ 2\rho q \left[\tan \theta - \tan(\theta - \alpha)\right]^{-1} \quad (45)$$

The gas flow work, i.e. the total power input into the same current strip is given as the braking Lorentz force in the flow direction $I_e Bd$ multiplied by the flow velocity $V$:
The generator efficiency \( \eta \) is defined in accordance with
turbine practice as

\[
\eta = \frac{P_e}{P_f} = \frac{-\varphi_k}{(VBd)} = \frac{x + d\cot\alpha + 2\rho q}{x + d\cot\alpha + 2\rho q} \quad (47)
\]

proving that this efficiency is identical to the external, i.e. the average, load factor \(-\varphi_k/(VBd)\).

8. Best positions of segmented generator electrodes

Two electrodes having total internal current connection are considered. Their best relative position will be defined as that which for a prescribed efficiency according to Eq. (47) yields the highest power output, Eq. (45).

The efficiency being known, the position \( x_0 \) can be written as a function only of the load angle \( \alpha \). When this expression for \( x_0 \) is inserted in Eq. (45) it is readily found that the problem of maximum power output is identical to that of finding the minimum of the internal resistance as given by Eq. (42). The relevant extrema are obtained for that value of \( \alpha \) which satisfies the equation

\[
\frac{d\Delta_y}{d\alpha} = 1 + \left(\frac{d}{\rho} + 2\Delta_y\right)\tan(\theta - \alpha) \quad (48)
\]
Using Eq. (27) the derivative of the shift $\Delta_v$ can be expressed as $1 - 2\Delta_u - b$. Eq. (48) then becomes

$$2\Delta_u + b + (d/\rho + 2\Delta_v)\tan(\theta - \alpha) = 0 \quad (48a)$$

and it expresses the condition for the two electrodes to have exactly opposed positions. This fact is easily proved when considering Fig. 3. The condition for two internally connected electrodes to be geometrically opposed is simply $x_1 = \rho b/2$. By inserting this value for $x_1$ in Eq. (34) an equation identical to (48a) is obtained.

The important Eq. (48a) was first deduced by Sutton (5) who showed that it represents the condition for current flow between two opposed electrodes.

9. Power, efficiency and internal resistance of the segmented generator with opposed electrodes

The expression for the generator efficiency is simplified in the case of the best electrode position when Eq. (47) is combined with Eq. (48a) to give

$$\eta = x_o (x_o + L\cos\theta / \sin\alpha)^{-1} \quad (49)$$

where the coordinate $x_o$ depends on the external load resistance. The length $L$ is given by Eq. (40). Using Eqs. (48a) and (49) the equation for the power output can be written

$$P^*_e = P_e / (2\rho \sigma_1 V^2 B^2 d) = \eta (1 - \eta) d \cos(\theta - \alpha)/L \quad (50)$$

Applying Eq. (42) the relation between power output, efficiency and internal resistance can be found:
\[ P_e = (VBd)^2 \eta (1 - \eta)/R_i \] (50a)

proving that the finite electrode size effects can be attributed to an increase of the internal resistance.

The figures 4-6 show the normalized power output \( P_e^* \) as a function of the efficiency \( \eta \) for three values of the Hall angle \( \theta \). The top curve of each diagram represents infinitely fine segmentation. These curves are already given in ref. (1). The lower pairs of curves give the influence of the degree of segmentation \( \rho/d \) and, where the clearness of the figure permits, they show the effect of changing from equal area electrodes and insulators, \( b = 1 \), to electrodes having one third of the insulator area, \( b = \frac{1}{3} \). The current flow in the central portion of the channel will deviate by the angle \( \delta \) from a direction perpendicular to the channel axis.

Fig. 7 shows the normalized internal resistance between two opposed electrodes when the segmentation degree \( \rho/d \) is equal to 0.1. \( R_i^* \) is related to the resistance \( R_i \), Eq. (42), as

\[
R_i^* = R_i 2 \rho \sigma_i / d = \\
= (1 + 2 \rho \Delta \nu / d)[\cos^2(\theta - \alpha) - 2 \rho \sin \alpha \cos \theta \cos(\theta - \alpha)]^{-1} \quad (51)
\]

and it is given as a function of the scaled insulator length \( b \). Eq. (48a) has been used in order to make \( R_i^* \) dependent only on \( \theta \) and \( b \). The dotted parts of the curves represent values of \( b \) for which the inequality (24) is not satisfied. The lower limit of this inequality is identical to that value of \( b \), Eq. (28), for which the shift \( \Delta \nu \) as well as the internal resistance have minima. Physically, this value of \( b \) represents the smallest length which the insulator can attain before current starts to flow between adjacent electrodes.
Because Eq. (51) gives a very simple relation between segmentation degree $p/d$ and normalized internal resistance $R^*$, the latter is shown only for $p/d = 0.1$. It is seen that the internal resistance is not critically dependent on the electrode-insulator length ratio. Generally, for transverse channel current flow in gases with Hall parameters of the order unity and larger, the equal insulator and electrode case gives practically the best generator performance.

10. The single load generator

In accordance with ref. (1) and (6) a Hall generator will be considered here as a MHD power generating duct with geometrically opposed and short-circuited pairs of electrodes, i.e. $x_o = 0$. This will be shown to represent a rather unfavourable special case of axial power extraction generators. The single external load is connected between two terminal electrodes situated at the channel entrance and exit.

The component of the static electric field in the flow direction is identical to the $x$-component of the moving frame field. Eq. (31) then gives the voltage $\varphi_x$ between two points situated at the channel center axis and separated by a distance $2R$:

$$\varphi_x = 2pA \sin \alpha = 2p VBd [x_o + d \cot \alpha + 2pq]^{-1}$$

(52)

The power output from a length $2R$ of the channel is still given by Eq. (45) from which the total axial current $I_x$ is found

$$I_x = \frac{P_e}{\varphi_x} = \sigma V Bd [x_o + d \tan (\theta - \alpha) + 2pq] \cdot [x_o + d \cot \alpha + 2pq]^{-1} [\tan \theta - \tan (\theta - \omega)]^{-1}$$

(53)
\(I_x\) is the algebraic sum of the axial component of the gas current and the current in a short-circuiting connection.

The axial voltage \(\varphi_l\) attains its open circuit value \(\varphi_{l0}\) when \(I_x = 0:\)

\[
\varphi_{l0} = 2\rho BVd \left\{ \left( d + 2\rho \Delta \nu \right) \left[ \cot \alpha - \tan(\theta - \alpha) \right] - 2\rho \right\}^{-1}
\]  

(54)

A load factor defined as \(k_l = \varphi_l/\varphi_{l0}\) is related to the efficiency \(\eta\), Eq. (47), as

\[
k_l = 1 - \eta
\]  

(55)

Exactly the same reasoning as that which led to Eq. (48a) would also here prove that the highest power output at a given efficiency is obtained when the current flows between exactly opposed electrodes. The short-circuiting connections then provide the axial charge transportation. However, in contrast to the segmented generator, the efficiency or load factor does not depend only on the external load, i.e. a prescribed efficiency will place restrictions also on the relative position of the short-circuited electrodes.

The figures 8-10 show the finite electrode size effects on the performance of the Hall generator. Curves representing infinitely fine segmentation, \(\rho/d = 0\), are already given in ref. (1), and ref. (6) has given the same type of performance curves in the case \(\beta_v = 8\), \(b = 1\).

Fig. 11 shows the current-voltage characteristics of the single load generator for a few different Hall parameters and segmentation degrees. The position \(x_o\) of the short-circuited electrode is taken as one segmentation length \(2\rho\). Only the equal electrode-insulator case \(b = 1\) has been considered. The dimensionless variables \(I_x^*\)
and $U^*$ are related to the generator axial current $I_x^*$, Eq. (53), and axial voltage $Q_x^*$, Eq. (52), as

$$I_x^* = \frac{I_x}{(\sigma_1 V B d)}$$  \hspace{1cm} (56)

$$U^* = \frac{Q_x^*}{(2\rho V B)}$$  \hspace{1cm} (57)

so that the normalized power output from a length $2\rho$ of the channel, $P_{e}^*$, is obtained by the product $I^* U^*$. The curves prove that the short-circuiting action of the electrodes on the axial Hall field rapidly increases with larger Hall parameters and smaller axial currents.

11. Maximum power output from a single load channel

The expression for the power output, Eq. (45), is considered as a function of the short-circuiting electrode position $x_0$. Derivation proves that $P_e$ has its maximum when

$$x_0 = d \left[ \cot \omega - 2 \tan (\theta - \omega) \right] + 2\rho (q - 2p)$$  \hspace{1cm} (58)

and, as could be expected, this value of $x_0$ makes the efficiency, Eq. (47), independent of the load angle $\alpha$ and equal to $1/2$.

The figures 12-14 show the dimensionless maximum power output $P_{em}^*$ from a length $2\rho$ of the channel. Only the case $b = 1$ is considered. The normalization of $P_{em}$ has been performed in the same way as that used for $P_e$ in Eq. (50). As Eq. (58) relates each value of the load angle $\alpha$ to a certain position $x_0$, the power output curves could as well have been given as unique functions of $\alpha$ instead.

A few facts may be stated about the curves. First, at their maxima the Eq. (48a) is satisfied implying that current flows
between exactly opposed electrodes. The power output is then as much as that which a segmented generator can produce, which is shown in figures 4-6. Second, the maximum performance for \( \eta = 1/2 \) is strongly dependent on the quantity \( x_o/d \) which should be positive and of the order a few times the segmentation degree \( \rho/d \). The curves also prove that the Hall generator connection, \( x_o = 0 \), only applies well in the case \( \rho/d \to 0, \beta_e \to \infty \).

12. Generators with a finite number of electrodes

Hitherto only channels with infinite lengths have been treated. The question of the distribution of the homogeneous central current flow between adjacent electrodes does not then arise. Cases of a finite number of identical and externally connected electrodes will be considered in this section. Constant conditions in the gas flow at the channel central portion are still assumed. The calculations and the curves given in the previous sections here apply only in the singular cases when the generator gas current is divided between the electrodes in such a way that the zero divergence condition for the current in an arbitrary number of closed external connections and loads is satisfied.

It has been shown that the highest power output at a prescribed efficiency is obtained when total internal current connection is established between exactly opposed electrodes. The performance of such generators with one, two and three independent and identical external loads will be investigated with respect to efficiency and other properties for different segmentation degrees and Hall parameters. The three generator types are shown schematically in Fig. 15 and they are designated by the Roman numbers I, II and III. For comparison the Hall generator, denoted by H, has been included. Only cases where the Hall parameter is unity and larger will be considered. Further, the generator types to be treated have all nearly transverse current flow and Fig. 7 or Ref. (4) will then
prove that very little can be gained in generator performance by diverging from the equal electrode and insulator case \( b = 1 \).

Fig. 3 shows that the condition for total gas current connection between two electrodes can be written

\[ x_0 = 2x_1 - b \rho + n2 \rho \quad , \quad n \text{ integer} \geq 0 \quad (59) \]

The coordinate \( x_1 \) is eliminated by the use of Eq. (34). As only geometrically opposed electrode positions are considered, the coordinate \( x_0 \) can be written

\[ x_0 = m2 \rho \quad , \quad m \text{ integer} \geq 0 \quad (60) \]

Combined Eqs. (34), (59) and (60) will yield the equation (61) below for the load angle \( \alpha \). This angle is contained not only in the trigonometrical factor but also in the shifts \( \Delta_u \) and \( \Delta_v \), Eq. (27).

\[ 2 \rho \Delta_u + b \rho + (d + 2 \rho \Delta_v) \tan(\theta - \alpha) = 2 \rho(n - m) \quad (61) \]

Upon inspection the following facts will be found:

- \( n = 0 \): the purely short-circuiting connection which leads to maximum flow work but no power output,

- \( n = 1, m = 0 \): the Hall generator, i.e. one load and short-circuited opposed electrodes,

- \( n = v \): the generator with \( v \) identical independent loads,

- \( n = m \): the generator with \( m \) loads and the gas current flow between opposed electrodes.

The Eq. (61) has been deduced from considerations of the current flow geometry. The same equation could as well have been obtained from the necessary potential relations which the different generator types must satisfy. Consider the axial voltage \( \phi_z \), given
by Eq. (52). For a generator with \( n \) loads, the electrode potential \( \varphi_k \), Eq. (47), is related to \( \varphi_l \) as

\[
n \varphi_l = - \varphi_k
\]  

(62)

Applying the Eqs. (52), (47), (37) and (60), Eq. (62) is transformed to an equation identical to (61).

13. Performance of generators with a finite number of electrodes

Table 1 gives the normalized power output \( P^* \), the flow work \( P^*_f \) normalized in the same way, the efficiency \( \eta = P^*_e/P^*_f \) and the dimensionless total axial current \( I^*_x \) for the four generator configurations shown in Fig. 15. Four values of the Hall parameter \( \beta_e \) and three segmentation ratios \( \rho/d \) have been considered. Each combination of generator configuration, Hall parameter and segmentation degree gives an equation of the type (61) for the pertinent load angle \( \alpha \). This equation being solved, the Eqs. (46), (45), (47) and (53) then yield the figures given in the table.

Eq. (47) shows that the efficiency \( \eta \) is identical to the normalized voltage between two internally connected electrodes. The terminal voltage \( U_t \) of a generator is therefore obtained as

\[
U_t = (\eta V B d) \times \text{(number of electrode pairs)/(number of loads)} \quad (63)
\]

The total gas flow power \( P_{ft} \) in the generator can be obtained from the table and Eq. (46) as

\[
P_{ft} = \sigma_l V^2 B^2 c d P^*_f \times \text{(generator length)} \quad (64)
\]

where the length \( c \) is the channel height in the magnetic field direction. As stated in section 1, this length was simply taken to be unity in the previous sections.
The current in an external load, \( I \), is related to the \( I^* \) of the table as

\[
I = \sigma_1 V B c d I^*/(\text{number of loads}).
\] (65)

The table shows that the same gas flow power is exerted in a Hall generator as in the generator I but that the power output from the Hall generator is always smaller, i.e., the Hall generator will give the same current but its terminal voltage will be smaller. For a large segmentation ratio and a not very high value of the Hall parameter, the difference is appreciable.

As stated above, the load angle \( \alpha \) will attain a specific value when a certain generator configuration with a finite number of electrodes is combined with prescribed values of \( \rho/d \) and \( \beta_e \). It is important to note then that the performance of any of the generators I, II, and III is not different from that of a similar segmented generator having independent loads and opposed electrodes and operating at the same load angle, i.e., efficiency. The table gives some indications of how the generator physical properties \( \beta_e \) and \( \rho/d \) have to be selected and combined in order to yield acceptable electrical performance of a generator having one, two or three external loads. E.g., low values of the Hall parameter will require a rather large size of the electrode segments in contrast to a sometimes expressed belief that a fine degree of segmentation is always favourable.

The largest flow power is exerted in generator I which gives the highest terminal voltage but suffers from low efficiency. At the expense of an increased number of loads, the efficiency will generally be higher and usually also the total power output except when the value \( \eta = 1/2 \) is exceeded. The difference in performance between generators I and II is appreciable, the same kind of difference between II and III being less pronounced. It has been pointed out (8)
that the comparatively low figures of generator I may not be serious as an actual MHD generator will probably require at least two loads, one load being its own magnet system.

**Acknowledgements**

The author is grateful to Drs. Josef Braun and Stig Lundquist for their kind and encouraging interest in this work. The computer calculations were performed by Mr. Stanley Svensson. Mr. Kjell Ericsson assisted at the graphical presentation of the computer results.
References

1. HARRIS L P and COBINE J D
   The significance of the Hall effect for three MHD generator configurations.

2. KERREBROCK J L
   Segmented electrode losses in MHD generators with nonequilibrium ionization.
   (BSD-TDR-64-35).

3. HURWITZ Jr H, KILB R W and SUTTON G W
   Influence of tensor conductivity on current distribution in a MHD generator.

4. WITALIS E A
   Performance of a segmented electrode MHD generator for various electrode-insulator length ratios.
   To be published in J. Nucl. Energy, Part C.

5. SUTTON G W
   Segmented generator with opposed electrodes.
   Private communication.

6. DZUNG L S
   The MHD generator in cross-connection.

7. ROSA R J
   Hall and ion-slip effects in a nonuniform gas.

8. BRAUN J
   Private communication.
Note added in proof

After the present work was completed, it was brought to the author's attention that Dr. L.S. Dzung of Brown, Boveri and Co, Baden, Switzerland, has also treated the problems of favourable electrode-insulator length ratios and electrode positions, however, without obtaining the explicit expressions, Eqs. (28) and (48a) for minimum generator internal resistance and best positions of electrodes respectively.

Dr. Dzung has kindly pointed out to the author that the treatment used here and in ref. 4 also applies when internal current leakage occurs between adjacent electrodes.

Dr. Dzung's analysis is scheduled to appear in the August 1965 issue of the Brown Boveri Review.
<table>
<thead>
<tr>
<th>( \beta_e = 10^* )</th>
<th>( \theta/d = .025 )</th>
<th>( \theta/d = .05 )</th>
<th>( \theta/d = .1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>( p_e^* )</td>
<td>( p_r^* )</td>
<td>( \eta )</td>
</tr>
<tr>
<td>I</td>
<td>( p_e^* )</td>
<td>( p_r^* )</td>
<td>( \eta )</td>
</tr>
<tr>
<td>II</td>
<td>( p_e^* )</td>
<td>( p_r^* )</td>
<td>( \eta )</td>
</tr>
<tr>
<td>III</td>
<td>( p_e^* )</td>
<td>( p_r^* )</td>
<td>( \eta )</td>
</tr>
<tr>
<td>( \beta_e = 24^* )</td>
<td>( p_e^* )</td>
<td>( p_r^* )</td>
<td>( \eta )</td>
</tr>
<tr>
<td>( \beta_e = 5.7^* )</td>
<td>( p_e^* )</td>
<td>( p_r^* )</td>
<td>( \eta )</td>
</tr>
<tr>
<td>( \beta_e = 11.4^* )</td>
<td>( p_e^* )</td>
<td>( p_r^* )</td>
<td>( \eta )</td>
</tr>
</tbody>
</table>

TABLE 1
Fig. 1. Geometry of the MHD channel.

Fig. 2. The Schwarz-Christoffel transformation.

Fig. 3. The conformal mapping of the MHD channel.
Fig. 4 Power-efficiency curves for the segmented generator with opposed electrodes.

Fig. 5 Power-efficiency curves for the segmented generator with opposed electrodes.
Fig. 6 Power-efficiency curves for the segmented generator with opposed electrodes.

Fig. 7 The normalized internal resistance of the segmented generator as function of the scaled insulator length.
Fig. 8 Power-efficiency curves for the Hall generator.

Fig. 9 Power-efficiency curves for the Hall generator.
Fig. 10 Power-efficiency curves for the Hall generator.

Fig. 11 Current-voltage relations for the single load generator.
Fig. 12 Maximum power output of the single load generator.

Fig. 13 Maximum power output of the single load generator.
Fig. 14 Maximum power output of the single load generator.

Fig. 15 Four generator configurations described in text.
Cross section measurements of the s«Ni(n, 120. The thermox process. By O. Tjälldin. 1963. 38 p. Sw. cr. 8:—.
115. Flame photometric determination of lithium contents down to 10—100 ppm in water samples. By O. Hellström and E. Aalto. 1964. 21 p. Sw. cr. 8:—.
117. Determination of the absolute disintegration rate of Cs+ sources by the tracer method. S. Hallström and D. Brune. 1963. 17 p. Sw. cr. 8:—.
118. Measurements of small exposures of gamma radiation with Cs137 and Cs137-Mn 60/Co thermoluminescence dosemeters. By B. Bjørgard, 1963. 20 p. Sw. cr. 8:—.
121. A gas target with a tritium gas handling system. By B. Holmqvist and T. Wiedling. 1964. 36 p. Sw. cr. 8:—.
122. The planning of a small pilot plant for development work on aqueous Cu-Ge alloys. By H. P. Myers and R. Westin. 1963. 7 p. Sw. cr. 8:—.
124. Determination of magnesium in needle biopsy samples of muscle tissue by means of neutron activation analysis. By D. Brune and H. E. Sjöberg. 1964. 8 p. Sw. cr. 8:—.
125. Absolute El transition probabilities in the dofermed nuclei Yb 169 and 171. By S. Hemqvist and E. Witalis. 1965. 10 p. Sw. cr. 8:—.
127. A Monte Carlo method for the analysis of gamma radiation transport through the International Documentation Center, Tumba, Sweden.
134. A study of the angular distributions of neutrons from the Be6(p,n)B7 reaction at low proton energies. By B. Antolovic', B. Holmqvist and T. Wiedling. 1964. 19 p. Sw. cr. 8:—.
135. A simple apparatus for fast ion exchange separations. By K. Samsahl. 1964. 6 p. Sw. cr. 8:—.
136. Determination of the neutron energy range 2.3—3.8 MeV. By A. Lauber and S. Molsmark. 1964. 28 p. Sw. cr. 8:—.