A short scan helical FDK cone beam algorithm based on surfaces satisfying the Tuy’s condition

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Abstract -- FDK method is the most popular cone beam algorithm to date. Traditionally, short scan helical FDK algorithms have been implemented based on horizontal transaxial slices. However, not every point on the horizontal transaxial slice satisfies Tuy’s condition for the corresponding \(\pi + \) fan angle segment of helix, which means that some points on the horizontal slices are incompletely sampled and are impossible to be exactly reconstructed. In this paper, we propose and implement an improved short scan helical cone beam FDK algorithm based on nutating curved surfaces satisfying the Tuy’s condition. This surface is defined by averaging PI surfaces emanating the initial and final source points of a \((\pi + \) fan angle\) segment of helix. One of the key characteristics of the surface is that every point on it satisfies the Tuy’s condition for the corresponding \((\pi + \) fan angle\) segment of helix, which means that we can potentially reconstruct every point on the surface exactly. This difference makes the proposed algorithm deliver a better-reconstructed image quality while requiring a smaller detector area than that of traditional FDK methods based on horizontal transaxial slices. Another characteristic of the proposed surface is that every point in the object space belongs to one and only one such surface. Therefore, the location of the short scan segment for reconstruction of a point in Cartesian coordinate can be pre-calculated and stored in a look up table. This enables us to perform reconstruction directly on rectangular grids. We compare the performance of the improved FDK algorithm with that of a quasi-exact algorithm based on data combination technique. The simulation results show that the reconstructed image quality of these two methods is about the same. We also provide a qualitative analysis of the link between the improved FDK and exact methods. The computational requirement of the proposed algorithm is the same as that of the traditional FDK method. We validate the proposed algorithm with a disc phantom.

I. INTRODUCTION

Cone beam tomography has been under active investigation for both clinical and small animal imaging. In clinical applications, development of cone beam algorithms is necessitated by the introduction of multi-row spiral scanner and the utilization of flat panel detectors in some tomographic systems. In micro CT, cone beam geometries are usually applied to improve the spatial resolution and sensitivity for small animal imaging.

The trend in clinical helical CT has been toward more detector rows (currently, 16), leading naturally to cone beam geometry. In order to account properly for the cone beam effects, numerous exact/quasi-exact and approximate reconstruction algorithms have been developed. They can be classified into two categories: Katsevich’s type [1] and Grangeat’s type [2]. Katsevich’s algorithm has drawn great interest recently since it only needs one dimensional shift invariant filtering and is theoretically exact. Researchers also developed various exact/quasi-exact algorithms [3-5] based on Grangeat’s theory that makes a link between the cone beam projection and 3D Radon derivatives. To our knowledge, the most practical algorithm based on Grangeat’s theory is developed in [4] since it only needs data collected within the Tam window [6] and the algorithm is of filtered backprojection structure, which is very desirable in practical application. Initially researchers thought that the algorithm developed in [4] cannot be applied for long object imaging since the data contamination problem. Complicated algorithms have been developed to address the so called long object problem for algorithms based on Grangeat’s theory [6-9]. However, it has been shown in [10] that as long as we define a short object bounded by PI lines emanating from the initial and final source points of a helical segment, the long object problem does not exist and the theory implied in [4] alone is enough to perform long object imaging. Although exact methods can deliver good reconstructed image quality theoretically, currently, they still possess the following common shortcomings: incapability of dealing with redundant or missing data, limited adaptation to variable table feed, rather complicated implementation and computational inefficient, all of which are key issues for practical clinical spiral CT. However, the importance of developing exact algorithms cannot be ignored since they can guide the design of accurate approximate methods. It is safe to say that if an approximate method delivers an accurate reconstruction result, then it has a close link to an exact method, although the link might not be found yet.

In contrast, approximate methods have more capability to cope with all kinds of practical situations but their reconstruction quality usually degrade as the cone angle increases. Many approximate methods exist and they can be classified into the following two types: one of rebinning type and the other of FDK type. The goal of rebinning type
algorithm is to convert the 3D tomography to 2D tomography. After the 2D data set is obtained, we can use the traditional 2D theory and the already available hardware to obtain the reconstruction fast. They are widely used in the current spiral clinical scanner since the cone angle involved is still very small. For example, the full cone angle is less than 2 degrees for the 16-slice (1mm/slice) clinical CT. However, they usually fail at medium or large cone angles. For the rebinning based algorithms, the better to fit a 2D rebinning plane to the 3D helix, then the more accurate the reconstruction results will be. This phenomenon can be observed from the development of single slice rebinning (SSR) [11], advanced single slice rebinning (ASSR) [12] and SMPR [13] algorithms.

FDK type algorithm is the most popular cone beam algorithm to date and was originally developed for circular orbit [14]. It was then generalized to helical trajectory by Wang et al [15]. FDK type algorithms can usually deliver a better result than rebinning based algorithms. Traditionally, helical FDK algorithm has been based on horizontal transaxial slices, and the ramp filtering was performed horizontally. Recently, Sourbelle and Kalender [16] improved the short scan FDK algorithm by performing the ramp filtering parallel to the tangent of the helix, just as the quasi-exact algorithm does. This modification improves the reconstruction quality greatly. But their method is still based on horizontal transaxial slices, and it can be easily shown that not every point in the horizontal transaxial slice satisfies Tuy’s condition for the corresponding PI + fan_angle segment of helix.

In this work, we further improve the short scan FDK algorithm by performing reconstructions on nutating curved surfaces satisfying the Tuy’s condition. The goal of long object helical cone beam tomographic imaging is to reconstruct a part of the object from the two dimensional projections collected over a segment of helix (Fig. 1). A segment of helix spanned from \( \lambda_{\text{min}} \) to \( \lambda_{\text{max}} \) rotates around an object can be described by:

\[
\hat{S}(\lambda) = (R \cos \lambda, R \sin \lambda, h \lambda) \quad \lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}]
\]  

(1)

where \( R \) is the source to iso-center distance, \( \lambda \) is the rotational angle of the helix and \( 2\pi h \) is the distance the x-ray source translates in one rotation.

Different algorithms have been proposed to obtain reconstructions from the axially truncated helical cone beam data. It has been shown that the minimal detector data needed to perform an exact helical cone beam reconstruction is that collected in the Tam window, which is bounded by projections of adjacent helical turns [3].

![Figure 1 Geometry of helical cone beam tomography.](image)

In this work, we propose a short scan helical FDK algorithm based on nutating curved surface satisfying the Tuy’s condition. Short scan helical FDK algorithm means reconstructing a slice from \( \pi + \alpha_{\text{max}} \) segment of helix. The fan angle \( \alpha_{\text{max}} \) is defined as follows:

\[
\alpha_{\text{max}} = 2\sin(r/R)
\]  

(2)

where \( r \) is the radius of the object support and \( R \) is the helix radius.

By using the property [7,17] that there is one and only one PI line passing each point in the object, it can be easily shown that for a segment of helix spanned from \( \lambda_{\text{min}} \) to \( \lambda_{\text{max}} \), the only region which satisfies the Tuy’s condition is that bounded by curved surface \( \Sigma_{\pi}(x,y,\lambda_{\text{max}}) \) and \( \Sigma_{\pi}(x,y,\lambda_{\text{min}}) \) (Fig. 1). \( \Sigma_{\pi}(x,y,\lambda_{\text{max}}) \) is formed by solid PI lines emitted from \( \lambda_{\text{max}} \) and \( \Sigma_{\pi}(x,y,\lambda_{\text{min}}) \) is formed by dashed PI lines emitted from \( \lambda_{\text{min}} \). The analytical form of \( \Sigma_{\pi}(x,y,\lambda_{\text{max}}) \) and \( \Sigma_{\pi}(x,y,\lambda_{\text{min}}) \) is derived in [10].

Traditionally, helical FDK algorithms are based on horizontal slices (Fig.2). These methods have one disadvantage: Not every point on the horizontal slice satisfies Tuy’s condition for the corresponding \( \pi + \alpha_{\text{max}} \) segment of helix used to reconstruct that slice, which means that not every point on the horizontal slices is completely sampled by projections collected over the corresponding \( \pi + \alpha_{\text{max}} \) segment of helix.
In this work, we based our FDK reconstruction on the averaged surface of $\Sigma(x,y)$ and $\Sigma_y(x,y)$. Since the averaged surface (the shaded surface in Fig.3) is between $\Sigma(x,y)$ and $\Sigma_y(x,y)$, every point on it satisfies Tuy's condition and can be potentially exactly reconstructed.

$$\lambda_{\text{min}} = -\frac{\pi}{2} - \frac{\alpha_{\text{max}}}{2}$$

Fig. 2 Geometry of short scan helical FDK algorithms based on horizontal transaxial slices

$$\lambda_{\text{max}} = \frac{\pi}{2} + \frac{\alpha_{\text{max}}}{2}$$

Fig. 3 Geometry of the proposed short scan helical FDK algorithms based on nutating curved surfaces

We describe the proposed algorithm based on Fig. 3. We only demonstrate how to reconstruct one single nutated curved surface with $\pi + \alpha_{\text{max}}$ segment of helix and other nutated surfaces can be reconstructed similarly. The X ray source goes from $\lambda_{\text{min}}$ to $\lambda_{\text{max}}$ along the helix. Assume $\lambda_{\text{min}} = -\pi/2 - \alpha_{\text{max}}/2$ and $\lambda_{\text{max}} = \pi/2 + \alpha_{\text{max}}/2$, $\alpha_{\text{max}}$ is the full fan angle and defined in equation 2.

The proposed short scan helical FDK algorithm for reconstructing a single curved surface is described as follows:

**Standard cone beam weighting**

$$g_u(u,v,\lambda) = \frac{R}{\sqrt{R^2 + u^2 + v^2}} g(u,v,\lambda)$$ (3)

where $g(u,v,\lambda)$ is the original projection, $g_u(u,v,\lambda)$ is the weighted projection, $R$ is the source to iso-center distance and $(u,v)$ is the detector coordinate system with $u$ placed horizontally. The detector is normalized to the iso-center.

**Parker weighting**

$$g_{\text{parker}}(u,v,\lambda) = g_u(u,v,\lambda) w_{\text{parker}}(\lambda, u)$$ (4)

where $g_{\text{parker}}(u,v,\lambda)$ is the Parker weighted projection. $w_{\text{parker}}(\lambda, u)$ is the Parker weight defined in [18] for short scan fan beam algorithms. If a helical segment of more than a half scan is to be used, then the weighting scheme proposed by [19] can be used.

**Ramp filtering**

$$g_r(u,v,\lambda) = \int dv' g_u(u,v',\lambda) k(u - u')$$ (5)

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \cos \eta & \sin \eta \\ -\sin \eta & \cos \eta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

(6)

where $\eta = \text{atan}(h/R)$, $k$ is the ramp filtering kernel and $u_i$ is parallel to the helix tangent.

**Cone beam backprojection**

$$f(\tilde{r}) = \frac{1}{2} \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} d\lambda \frac{R^2}{(\tilde{r} - s(\lambda)) \cdot \tilde{t}_u} g_r(u(\tilde{r}), v(\tilde{r}), \lambda)$$ (7)

where $\tilde{r} = (x,y,z(x,y))$ is a point in the object, $\tilde{t}_u$ is the unit vector pointing to the detector center from the source. $u(\tilde{r})$ and $v(\tilde{r})$ are the cone beam projections of $\tilde{r}$ into the detector space and $z(x,y)$ can be calculated in the following equation

$$z(x,y) = \frac{\Sigma_z(x,y,\lambda_{\text{max}}) + \Sigma_z(x,y,\lambda_{\text{min}})}{2}$$ (8)

After we obtain the reconstructions on many curved surfaces corresponding to different helical segments, we finally do a linear interpolation along the z direction so that we can obtain reconstructions on rectangular grids.

Actually, It can be shown that every point in the object support belongs to one and only one shaded surface shown in Fig. 3. Therefore, in the implementation, we can calculate the location of short scan segment for every object point on Cartesian grids and put them in a lookup table. In this way,
we do not have to do z interpolation for the final presentation of the reconstructed images.

III. Simulation Results

A disk phantom[4] is scanned and reconstructed. The scanning parameters are shown in Table 1. The detector is normalized to the iso-center in the simulation.

<table>
<thead>
<tr>
<th>Scan parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source to center distance</td>
<td>36 cm</td>
</tr>
<tr>
<td>Detector dimension</td>
<td>256 × 256 cm</td>
</tr>
<tr>
<td>Detector size</td>
<td>22.528 × 22.528 cm</td>
</tr>
<tr>
<td>Sampling distance</td>
<td>0.088 cm</td>
</tr>
<tr>
<td>Length of helix segment</td>
<td>2 turns</td>
</tr>
<tr>
<td># of proj/s per rotation</td>
<td>360</td>
</tr>
<tr>
<td>Helical Pitch</td>
<td>10 cm</td>
</tr>
<tr>
<td>Object support radius</td>
<td>11.264 cm</td>
</tr>
<tr>
<td>Effective cone angle</td>
<td>16 degrees</td>
</tr>
</tbody>
</table>

Table 1 Scanning parameters

Four reconstruction methods are implemented. For methods 1-3, the ramp filter is apodized by a Shepp Logan window.

**Method 1:** Short scan FDK, ramp filtering in horizontal direction and reconstruction on horizontal slices [20]

**Method 2:** Short scan FDK, ramp filtering in the direction of helix tangent and reconstruction on horizontal slices [16]

**Method 3:** The proposed short scan algorithm in this work

**Method 4:** Quasi exact reconstruction [21]

Fig. 4 shows the result for the disk phantom (Display window [0.7 1.3]). Severe streak artifacts are evident in reconstruction (a) because the filtering is performed along the horizontal direction and the reconstruction is based on horizontal slices. After the filtering is performed along the helix tangent, the streak artifacts are greatly reduced, but they still can be observed. (c) is the result by the proposed algorithm with very little streak artifact. Its image quality is similar to that of (d), which is the result of quasi-exact reconstruction.

Different reconstruction algorithms require different detector areas. The proposed algorithm requires a less detector area than that of traditional FDK methods based on horizontal slices. For FDK type algorithms, the detector should be big enough so that when the source is at $\lambda_{\min}$ or $\lambda_{\max}$, there is still enough projection data for filtering and backprojection. And for the quasi-exact algorithm, it is well known that the necessary data is those within the Tam window.

Fig. 4 Reconstructed sagittal image (y=0) of the disk phantom (display window [0.7 1.3]). (a) Result of method 1 (b) Result of method 2 (c) Result of method 3 (d) Result of method 4

IV. Discussion & Conclusion

In this paper, we proposed a novel short scan helical FDK algorithm based on nutating curved surfaces satisfying the Tuy’s condition. One of the key characteristics of the curved surface is that every point on it satisfies the Tuy’s condition. The filtering is along helix tangent, just as in the quasi-exact reconstruction methods [4]. Although filtering along helix tangent can deliver a better reconstruction result than filtering along horizontal method, it is still not an exact filter since second intersection artifacts are not accounted for in the ramp filter along one single direction. The exact filtering can only be achieved by filtering along different directions [1,22].

Compared to the traditional FDK algorithms based on horizontal slice, the proposed algorithm can deliver a better image quality while requires less detector area. There is a connection between the helical FDK algorithm with the exact algorithm considering the link between the ramp filtering and shift variant filtering made by [23,24]. The only reason that the result of shift variant filtering for quasi-exact algorithm proposed in [4] is unbounded in the axial direction is that we have to cope with the truncation problem and use the data combination technique to acquire the Radon data over the whole plane. If the truncation problem is ignored, then the quasi-exact filter will reduce to ramp filtering along the tangent direction, which is the filtering method we use in this work.
Although a rigorous theoretical derivation is not provided for how accurate the proposed method can deliver, we believe that its reconstructed image quality is close to that delivered by exact methods. In Figure 3, if we only have a short object bounded by $\sum_{t}(x,y,\lambda_{\text{max}})$ and $\sum_{t}(x,y,\lambda_{\text{min}})$, then by applying appropriate weighting and filtering, the short object can be exactly reconstructed if non-truncated projections are collected from $\lambda_{\text{min}}$ to $\lambda_{\text{max}}$. In this work, we only provide heuristic filtering and weighting schemes. Therefore, the inexactness of the proposed FDK algorithm comes from three aspects. First of all, the filtering is still not exact. For the helical scanning, exact filtering should be along different directions in order to eliminate the second intersection artifacts. Secondly, the parker weighting might not be the exact weighting for the proposed geometry. Thirdly, reconstruction on the surface may receive minimal contamination from the region outside of $\sum_{t}(x,y,\lambda_{\text{max}})$ and $\sum_{t}(x,y,\lambda_{\text{min}})$. This contamination is analyzed in [10] and can be minimized by reconstructing on the surface which is an average of $\sum_{t}(x,y,\lambda_{\text{max}})$ and $\sum_{t}(x,y,\lambda_{\text{min}})$.

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VI. REFERENCES