Using Image Theory for Finite Electrostatic Problems –
Some Observations and Guidelines

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Abstract: The following question is examined: can image theory be used to provide reasonably accurate results for electrostatic problems involving finite conducting structures? The moment method is used in this investigation to determine the surface charge distribution induced on a finite conducting plane by a point charge above the plane. The size of the plane and position of the charge are arbitrary. Results are compared with those obtained from image theory for various point charge positions. These results can provide guidelines regarding when it is appropriate to use image theory to obtain reasonably accurate results for electrostatic problems involving finite conducting structures.

INTRODUCTION

Recent papers [1-5] have discussed the validity of using image theory to determine the radiation properties of printed circuit boards of finite dimension. Although not strictly valid, it has been shown [3,4] that image theory can be used to provide accurate results for problems involving time-varying currents that are electrically close to finite planes which have dimensions greater than one wavelength. Although these results provide helpful insight for problems involving high frequency currents, the required dimensions for static problems are still infinite. In this paper, a related question is addressed: can image theory be used to provide reasonably accurate results for electrostatic problems involving finite ground structures? The answer to this question can provide guidance to those attempting to model electrostatic phenomena.

In an effort to obtain insight into this question, the moment method is used to determine \( \rho_s \), the charge distribution induced on an infinitesimally thin finite conducting plate by a point charge placed above the plate. The results obtained from the moment method analysis are compared to those obtained using image theory, yielding helpful guidelines to know when image theory can be used to provide reasonably accurate results for electrostatic problems involving finite ground structures. After a brief review of the moment method, image theory, and the error analysis used, results will be provided, followed by conclusions.

METHOD OF ANALYSIS

Computation Using the Moment Method

Originally introduced by Harrington [6,7], the moment method has been used in the analysis of a large variety of electromagnetics problems. For this particular problem, we assume that a point charge \( Q \) is located at some position \((x_p, y_p, h)\) above a perfectly conducting square plate, 1 m on each side. As shown in Figure 1, this plate is in the \( z = 0 \) plane, centered about the origin.

![Figure 1. Point Charge Above Conducting Plate](image_url)

The potential \( V_0 \) at any point \((x, y, 0)\) on plate is given as

\[
V_0 = \int \int \frac{\rho_s ds}{4\pi\epsilon_0 r} + \frac{Q}{4\pi\epsilon_0 r_p}
\]

(1)

where \( \rho_s \) is the unknown charge distribution on the plate, \( r \) is the distance from an increment of that charge distribution to the field point \((x, y, 0)\) and \( r_p \) is the distance from the point charge to the field point. Employing the moment method [6-8], a matrix equation is derived from which the unknown charge distribution \( \rho_s \) is determined. In particular, the finite conducting plate shown in Figure 1 is divided into \( N \) subsections. Assuming that the unknown charge distribution is constant over each subsection and using point matching (i.e., using pulse functions as basis
functions and delta functions as weighting functions) the following matrix equation is obtained [8]

$$[A][\rho] = [g].$$  

Typical matrix elements are

$$A_{ij} = \int \int \frac{dx'dy'}{r_{ij}},$$

and $$\rho_i$$ is the unknown surface charge distribution on the $$i^{th}$$ subsection. In the above equations, $$r_{ij}$$ is the distance from the center of the $$i^{th}$$ subsection (which is the field point) to the point $$(z', y', 0)$$, and is given as

$$r_{ij} = \sqrt{(x_i - x')^2 + (y_i - y')^2}.$$  

Similarly, $$r_{ip}$$ is the distance from the center of the $$i^{th}$$ subsection to the point charge, and is given as

$$r_{ip} = \sqrt{(x_i - x_p)^2 + (y_i - y_p)^2 + h^2}.$$  

The integration for $$A_{ij}$$ is carried out over the $$j^{th}$$ subsection, and is written as

$$A_{ij} = \int_{y_{j1}}^{y_{j2}} \int_{x_{j1}}^{x_{j2}} \frac{dx'dy'}{r_{ij}},$$

where $$(x_{j1}, y_{j1})$$ and $$(x_{j2}, y_{j2})$$ define the two corners of that subsection. The elements $$A_{ij}$$ can be computed numerically, or via the closed form expressions provided in the Appendix. The matrix equation is then solved for the unknown surface charge distribution.

**Computation Using Image Theory**

In order to address the question regarding the validity of using image theory for finite conducting structures, the surface charge distribution is also determined using image theory. In particular, since the coordinates of the point charge are $$(x_p, y_p, h)$$, the surface charge density induced at the point $$(x, y, 0)$$ on the conducting plane is given as

$$\rho_{si}(x, y) = \frac{-Qh}{2\pi \epsilon_o \left[(x - x_p)^2 + (y - y_p)^2 + h^2\right]^{3/2}}.$$  

**Error Analysis**

Two methods will be used to compare the surface charge distributions obtained from the moment method and image theory. First of all, the charge distributions will be plotted and visually inspected. As the point charge moves away from the finite plate, the difference in the charge distributions will be quite obvious. The second method of comparison will be to calculate a normalized error term for each case. In particular, a normalized least squares error will be determined. If $$\rho_s(x_i, y_i)$$ denotes the surface charge density at $$(x_i, y_i, 0)$$ calculated using the moment method, and $$\rho_{si}(x_i, y_i)$$ denotes the surface charge density at $$(x_i, y_i, 0)$$ calculated using image theory, the following error term will be used:

$$E_{\text{norm}} = \frac{\sum_{i=1}^{N} \left| \rho_s(x_i, y_i) - \rho_{si}(x_i, y_i) \right|^2}{\sum_{i=1}^{N} \left| \rho_s(x_i, y_i) \right|^2}.$$  

**RESULTS**

Values of $$\rho_s$$ are determined for various charge locations. Although the conducting plate can have any desired dimensions, all plots provided in this paper are for a square plate, 1 m on each side, centered at the origin. The plate was assumed grounded, and a $$-1 \mu C$$ point charge was at some location above the plate.

**Point Charge Over the Center of the Plate**

Representative plots are given in Figures 2-4 of cases when the point charge is located directly over the center of the square plate.

**Figure 2.** $$\rho_s(\mu C/m^2)$$ when the charge is 5 cm above the center of the plate.

From these plots it can be observed that when the point charge is close to the conducting plate $$\rho_s$$ tends to be very concentrated and as the point charge is moved away from the plate the charge distribution tends to spread out over the plate. When this happens, an “edge effect” can be observed where the charge distribution increases at the edges of the plate. It is this edge effect that image theory does not predict. For instance, calculated values of $$\rho_{si}$$ are shown in Figures 5 and 6 for the cases when the point charge is 5
and 50 cm above the plate, respectively.

Figure 3. $\rho_s (\mu C/m^2)$ when the charge is 25 cm above the center of the plate.

Figure 4. $\rho_s (\mu C/m^2)$ when the charge is 50 cm above the center of the plate.

Figure 5. $\rho_{st} (\mu C/m^2)$ when the charge is 5 cm above the center of the plate.

Comparing Figure 2 with Figure 5, and Figure 4 with Figure 6 one readily observes that when the charge is close to the conducting plate the results obtained from the moment method and image theory are very similar, but as the charge moves away from the plate there is quite a discrepancy.

To further illustrate this point, the charge distributions obtained from the moment method and image theory for a particular “cut” on the ground plane (i.e., for $y = 0$) are plotted in Figures 7 and 8 for point charge heights of 5 and 50 cm. Again, it is observed that results obtained from image theory and the moment method are very similar if the point charge is “close” to the conducting plane, but are quite different as the charge moves away from the plate.

Figure 6. $\rho_{st} (\mu C/m^2)$ when the charge is 50 cm above the center of the plate.

Figure 7 Comparison of $\rho_s$ and $\rho_{st}$ when the charge is 5 cm above the center of the plate.

Figure 8 Comparison of $\rho_s$ and $\rho_{st}$ when the charge is 50 cm above the center of the plate.
To get a little better handle on how "close" the charge needs to be to the plate for one to feel some degree of comfort using image theory, a normalized least squares error is computed, as described above. Results for various point charge heights are shown in Table 1. The ratio \( h/l_{\text{min}} \) is also shown in Table 1, where \( h \) is the height of the point charge above the plate. If the point charge were projected onto the plate, \( l_{\text{min}} \) is minimum distance from the point charge's projected position to the edge of the plate.

Table 1. Normalized Error vs. Charge Height When Charge is Centered Over Plate.

<table>
<thead>
<tr>
<th>( h(m) )</th>
<th>( h/l_{\text{min}} )</th>
<th>( E_{\text{norm}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1</td>
<td>0.54</td>
</tr>
<tr>
<td>0.25</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>0.05</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1</td>
<td>0.02</td>
</tr>
<tr>
<td>0.025</td>
<td>0.05</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Point Charge At \((0.25, 0.25, h)\)

As the location of the point charge moves closer to the edge of the plate one would expect that the "edge effect" would again have an influence on the validity of using image theory. In order to investigate this, the point charge was again placed above a square plate (1 m on a side) - now at \((0.25, 0.25, h)\). Figures 9 and 10 show the calculated values of \( \rho_s \) for heights of 5 and 50 cm. These plots can be compared to Figures 11 and 12, which are the values of the surface charge distribution calculated with image theory.

![Figure 9](image9.png)

Figure 9. \( \rho_s(\mu C/m^2) \) when the charge is at \((0.25, 0.25, 0.05)\).

![Figure 10](image10.png)

Figure 10. \( \rho_s(\mu C/m^2) \) when the charge is at \((0.25, 0.25, 0.5)\).

To aid in the visual comparison, the values of \( \rho_s \) and \( \rho_{si} \) are also plotted versus \( z \) (for \( y = 0 \)) in Figures 13 and 14. Note that the two solutions diverge for the charge distribution near the edge of the plate. The normalized errors were calculated for these cases, and are listed in Table 2. Results compare favorably to when the charge was over the center of the ground plane. In particular, it appears that in both cases when the ratio of \( h/l_{\text{min}} \) is around 0.1 the normalized error falls to approximately 3%. It should be pointed out, however, that even though the error fell to approximately 3%, Figures 13 and 14 show that the charge distributions obtained from image theory and the moment method still give considerably different results near the edge of the plate.

![Figure 11](image11.png)

Figure 11. \( \rho_{si}(\mu C/m^2) \) when the charge is at \((0.25, 0.25, 0.05)\).

![Figure 12](image12.png)

Figure 12. \( \rho_{si}(\mu C/m^2) \) when the charge is at \((0.25, 0.25, 0.5)\).
finite conducting plate by a point charge has been investigated using the moment method and image theory. These charge distributions have been compared, and guidelines have been suggested regarding the accuracy of using image theory for electrostatic problems involving finite conducting structures.

**Moment Method APPENDIX**

It can be shown that for \( i \neq j \)

\[
A_{ij} = IA + IB
\]

where

\[
IA = B + bg(P1) - T \cdot \log(P2) + S \cdot \log(P3/P4) \tag{10}
\]

\[
IB = T \cdot \log(P5) - B \cdot \log(P6) - D \cdot \log(P7/P8) \tag{11}
\]

\[
B = Yj2 - yi, \tag{12}
\]

\[
T = Yj1 - yi, \tag{13}
\]

\[
S = xj2 - xi, \tag{14}
\]

\[
D = xj1 - xi. \tag{15}
\]

In addition

\[
P1 = S + E1, \tag{16}
\]

\[
P2 = S + F1, \tag{17}
\]

\[
P3 = D + E1, \tag{18}
\]

\[
P4 = T + F1, \tag{19}
\]

\[
P5 = D + F2, \tag{20}
\]

\[
P6 = D + E2, \tag{21}
\]

\[
P7 = B + E2, \tag{22}
\]

\[
P8 = T + F2, \tag{23}
\]

where

\[
E1 = \sqrt{D^2 + S^2}, \tag{24}
\]

**CONCLUSION**

Although image theory is commonly used when modeling problems involving finite conducting structures, the errors incurred in such cases can be significant. Authors in previous works have examined the conditions under which image theory can be used to provide reasonable results for cases involving high frequency electromagnetic fields. However, many problems of interest to EMC engineers involve static or quasi-static fields. The charge distribution induced on a

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**Figure 13** Comparison of \( \rho_s \) and \( \rho_{si} \) when the charge is at \((0.25,0.25,0.05)\).

**Figure 14** Comparison of \( \rho_s \) and \( \rho_{si} \) when the charge is at \((0.25,0.25,0.5)\).

**Table 2. Normalized Error vs. Charge Height. Charge is at \((0.25,0.25,h)\).**

<table>
<thead>
<tr>
<th>( h(m) )</th>
<th>( h/l_{\text{min}} )</th>
<th>( E_{\text{norm}} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>0.61</td>
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<td>0.25</td>
<td>1.0</td>
<td>0.38</td>
</tr>
<tr>
<td>0.125</td>
<td>0.5</td>
<td>0.15</td>
</tr>
<tr>
<td>0.05</td>
<td>0.2</td>
<td>0.03</td>
</tr>
<tr>
<td>0.025</td>
<td>0.1</td>
<td>0.03</td>
</tr>
</tbody>
</table>
\[ F_1 = \sqrt{T^2 + S^2}, \]  
\[ E_2 = \sqrt{B^2 + D^2}, \]  
\[ F_2 = \sqrt{T^2 + D^2}. \]

For \( i = j \), it can be shown that [7,8]

\[ A_{ij} = 2(\Delta y \cdot \log Z_1 + \Delta x \cdot \log Z_2) \]  
\[ \text{where} \]

\[ \Delta y = y_{j2} - y_{j1} \]  
\[ \Delta x = x_{j2} - x_{j1}, \]

\[ Z_1 = \frac{\Delta x}{\Delta y} + \sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2 + 1} \]

and

\[ Z_2 = \frac{\Delta y}{\Delta x} + \sqrt{\left(\frac{\Delta y}{\Delta x}\right)^2 + 1}. \]

REFERENCES


