

Design of a Helmholtz Coil For Susceptibility Testing using Variational Calculus and Experimental Verification

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Abstract : Helmholtz coils are used for high quality susceptibility testing. In this paper a mathematical model is developed which predicts optimum design . The model is compared to experimental results. The model uses a specific area of variational calculus called Lagrangian multipliers. In Lagrangian multipliers we minimize or maximize a function with constraints. We can maximize the Helmholtz resonance frequency with constraints on the magnetic field (magnetic flux density), the electric field, current and impedance. This model will tell us how to design Helmholtz coils to maximize the magnetic field, maximize the self resonance, minimize the electric field, and minimize the number of turns. This model will save designer and builder of Helmholtz coils much time and labor.

INTRODUCTION

AC Helmholtz coils can be used for high quality susceptibility testing. This paper describes the design of a Helmholtz coil that is a candidate test apparatus for low frequency magnetic field susceptibility testing. The use of the coil would reduce electromagnetic compatibility qualification test time, and subject the experiment under test to a relatively uniform field. The optimal design of Helmholtz coils depends upon maximizing some physical quantities and minimizing others. Hence the use of variational calculus is ideal for designing Helmholtz coils. The Helmholtz coils self resonance should be maximized, the magnetic field should be maximized, the self and mutual inductances should be minimized, the impedance should be minimized, and the electric field should be minimized. The variational principle that works best for Helmholtz coils is the use of Lagrangian multipliers [1]. There are various ways of approaching this problem. We could maximize the resonance frequency with the current and electrical field as constraints. We could minimize the current with the resonance frequency and electric field as constraints. One focus of this paper will be to maximize the self resonance frequency of the Helmholtz coils with the current and magnetic field as constraints.

All the physical quantities can be expressed as a function of number of turns and coil radius. The final predicted result will be the minimum number of turns for the maximum resonance for given values of the current, electrical field, magnetic field, impedance, and coil radius.

The basics of the model are as follows: We let F be the resonance frequency expression and we maximize it subject to constraint equations. We express the current I in terms of the magnetic field, the number of turns in the coils, and the radius of the coils. This equation becomes a constraint equation. The electric field can be another constraint equation, the impedance another. We express all physical quantities in terms of the number of turns and the coil radii. As an example let us maximize the coil self resonance with the current I in terms of the magnetic field as a constraint equation. The system of equations as a function of the number of turns N and the radius r becomes:

$$\frac{\partial F}{\partial N} + \lambda \frac{\partial I}{\partial N} = 0 \quad (1)$$

$$\frac{\partial F}{\partial r} + \lambda \frac{\partial I}{\partial r} = 0 \quad (2)$$

We solve these two equations simultaneously, and get an expression for the number of turns N in terms of magnetic field, and coil radius. We assign values of interest to the magnetic field, and coil radius. This result gives us the minimum number of turns that will maximize the resonance for a given coil radius and magnetic field.

Another example is to maximize the magnetic field with the coil self resonance as a constraint condition. This would require solving the simultaneous equations :

$$\frac{\partial B}{\partial N} + \lambda \frac{\partial F}{\partial N} = 0 \quad (3)$$

$$\frac{\partial B}{\partial r} + \lambda \frac{\partial F}{\partial r} = 0 \quad (4)$$

Still another example is to maximize the self resonance with the magnetic field as a constraint condition. This would require solving the simultaneous equations

$$\frac{\partial F}{\partial N} + \lambda \frac{\partial B}{\partial N} = 0 \quad (5)$$

$$\frac{\partial F}{\partial r} + \lambda \frac{\partial B}{\partial r} = 0 \quad (6)$$

Where B is the magnetic field or more specifically called magnetic flux density. Solving these two equations simultaneously we get the maximum magnetic field for a given number of turns, coil radius, and self resonance value. The variational principle will concentrate on the variational combination of equations (5) and (6).

The research paper will use the variational principle called Lagrange multipliers to maximize or minimize a physical quantity of the Helmholtz coils subjected to several constraint equations. This variational model applied to the Helmholtz coils can predict the design of a Helmholtz coils for susceptibility testing

The design of these coils is done with as few turns as possible to minimize series impedance, minimize the electric field, and maximize the resonance frequency. However, minimizing the number of turns means that the current must be maximized to maintain the desired ampere-turn product. The ampere-turn product is another quantity that can be used as a constraint equation in the variational principle. Since optimum design depends upon some physical quantities being maximized and others minimized. This research is a very flexible mathematical model for designing Helmholtz coils. Experimental data is compared to the model. The variational principle can guide any design work of the Helmholtz coils.

VARIATIONAL CALCULUS ON HELMHOLTZ COILS USING LAGRANGIAN MULTIPLIERS

The expression for the resonance frequency of the series connected AC Helmholtz coils is [2] and :

$$F = \frac{1}{2\pi\sqrt{(L+M)(C_0+C_G)}} \quad (7)$$

where:

F= resonant frequency, Hz.

L= external inductance of coil, Henrys

M= mutual inductance between the coil pair, Henrys

C₀= coil inter-winding capacitance, farads

C_G= coil capacitance to ground, farads

Where the mutual inductance is :

$$M = \alpha N^2 r \quad (8)$$

The series inductance can be estimated using the following expression derived by Ramo et al for the external inductance of a circular coil where the wire bundle cross section is circular in shape and small compared to the coil radius [3] :

$$L = N^2 r \mu_0 \left[\ln\left(\frac{16r}{a}\right) - 2 \right] \quad (9)$$

N = number of turns (same for each coil).

μ_0 = permeability of free space, Henrys/meter

$\alpha = 0.494 \times 10^{-6}$ Henrys/meter

r = coil radius, meters

a = diameter of wire bundle cross section, meters

Because the total inductance (and coil impedance) is directly proportional to the number of turns squared, the number of turns must be minimized. The importance of the resonance in Helmholtz coil calculations are stressed by Bronaugh[4].

Taking the derivative of the resonance frequency with respect to N yields:

$$\frac{\partial F}{\partial N} = \frac{-\left[2Nr\mu_0 \left(\ln\left(\frac{16r}{a}\right) - 2 \right) + 2\alpha Nr \right]}{\left[4 \left[\pi \left[N^2 r \mu_0 \left(\ln\left(\frac{16r}{a}\right) - 2 \right) + \alpha N^2 r \right] c_T \right]^{\frac{3}{2}} \right]} c_T \quad (10)$$

The derivative with respect to r is:

$$\frac{\partial F}{\partial r} = \frac{-\left[N^2 \mu_0 \left(\ln\left(\frac{16r}{a}\right) - 2 \right) + N^2 \mu_0 + \alpha N^2 \right]}{\left[4 \left[\pi \left[N^2 r \mu_0 \left(\ln\left(\frac{16r}{a}\right) - 2 \right) + \alpha N^2 r \right] c_T \right]^{\frac{3}{2}} \right]} c_T \quad (11)$$

One closed form solution for the magnetic flux density produced by a series-driven system of two identical coils [5] :

$$B_z = \frac{\mu_0 N I r^2}{2} \left[\frac{1}{\left(z^2 + r^2 \right)^{\frac{3}{2}}} + \frac{1}{\left(\left(z^2 + r^2 \right)^2 + r^2 \right)^{\frac{3}{2}}} \right] \quad (12)$$

where,

B_z = magnetic flux density, Teslas

I = current, Amperes

d = coil separation, meters

Z = distance along common axis, meters

Field uniformity can be computed by iterating the above equation over the axial distance z . For two circular coils separated by one radius, the magnetic field is relatively uniform throughout the cylindrical volume whose radius is approximately half the coil radius.

For a Helmholtz coil configuration, $d = r$. At the center of the test volume (i.e., $z = r/2$), (12) can be simplified:

$$B = \frac{(\gamma)NI}{r} \quad (13)$$

where

$$\gamma = 8.99 \times 10^{-7} \quad (14)$$

The derivative of the magnetic field with respect to r yields :

$$\frac{\partial B}{\partial r} = -\gamma N \frac{I}{r^2} \quad (15)$$

Now using the calculus of variations with the Lagrangian multiplier technique with the Lagrangian multiplier:

$$\frac{\partial F}{\partial N} + \lambda \frac{\partial B}{\partial N} = 0 \quad (16)$$

Performing the differentiations:

$$\begin{aligned} \frac{\partial F}{\partial N} + \lambda \frac{\partial B}{\partial N} &= \\ &-\left[2Nr\mu_0\left(\ln\left(16\frac{r}{a}\right)-2\right)+2\alpha Nr\right] \\ &\left[4\left[\pi\left[N^2r\mu_0\left(\ln\left(16\frac{r}{a}\right)-2\right)+\alpha N^2r\right]c_T\right]^{\frac{3}{2}}\right]^{c_T}+ \\ \lambda\left(\gamma\frac{1}{r}\right) &= 0 \end{aligned} \quad (17)$$

Solving for λ we obtain :

$$\lambda = \frac{r}{2[N]\left[\pi\left[\sqrt{N^2r(4\mu_0\ln(2)+\mu_0(\ln(r))}-\mu_0\ln(a)-2\mu_0+\alpha\right]\right]\right]} \quad (18)$$

Substituting λ into the following variational equation :

$$\frac{\partial F}{\partial r} + \lambda \frac{\partial B}{\partial r} = 0 \quad (19)$$

yields:

$$\begin{aligned} &\frac{\partial F}{\partial r} + \lambda \frac{\partial B}{\partial r} = 0 = \\ &\frac{-\left[N^2\mu_0\left(\ln\left(16\frac{r}{a}\right)-2\right)+N^2\mu_0+\alpha N^2\right]}{\left[4\left[\pi\left[N^2r\mu_0\left(\ln\left(16\frac{r}{a}\right)-2\right)+\alpha N^2r\right]c_T\right]^{\frac{3}{2}}\right]^{c_T}}+ \\ &\frac{r\left[-\gamma Br/\gamma\right]I/r^2}{\left[2\left[\frac{Br}{\gamma}\left[\pi\left[\sqrt{\frac{Br}{\gamma}}\right]^2r\left(\mu_0\ln\left(16\frac{r}{a}\right)-2\mu_0+\alpha\right)c_T(\gamma)\right]\right]\right]\right]} \end{aligned} \quad (20)$$

Solving for N , the number of turns, we obtain

$$N = \frac{\frac{1}{2}\left(\mu_0\ln\left(16\frac{r}{a}\right)\mu_0+\alpha\right)Br}{\left[I\left[\gamma\left(\mu_0\ln\left(16\frac{r}{a}\right)-2\mu_0+\alpha\right)\right]\right]} X$$

$$\sqrt{\frac{4\mu_0\ln(2)+\mu_0\ln(r)-\mu_0\ln(a)-2\mu_0+\alpha}{\left(\mu_0\ln\left(16\frac{r}{a}\right)-2\mu_0+\alpha\right)}} \quad (21)$$

The experimental data and theoretical results on the AC Helmholtz coils at 60 Hz are in the following table :

B	N	a	r	I	NO
10	20	.04215	.64	35.3	12.7
6.32	20	.04215	.64	22.3	12.5
10	16	.03731	.64	44.15	10

Where B is the magnetic field in Gauss, N is the number of coil turns, a is the coil bundle diameter in meters, r is the coil radius in meters, I is the current in amperes, and NO is the optimum number of turns predicted by the above variational Lagrange multiplier method with coil resonance frequency and magnetostatic fields as dependent variables. The experimental results above yield a magnetic field that has a corresponding frequency below the Helmholtz resonance frequency. However the number of turns in the experimental data was not optimum. The number of turns that yields a simultaneous optimum value of the magnetic field and the resonance frequency is obtained theoretically in the variational principle

model. The theoretical results predicted by the variational Lagrange multiplier technique will yield the optimum number of turns for the resonance frequency with the magnetic field as a constraint. This combination was a good candidate for the variational principle since the magnetic field increases with the number of coil turns and the resonance frequency decreases with the number of turns-hence there is an optimum number of turns for both quantities simultaneously. The optimal number of turns predictions will be improved when frequency dependent fields are used as explained in the section on future work below.

FUTURE WORK

Future work will focus on much more complicated variational analysis using frequency dependent magnetic and electric fields. The variational dimensions will be higher and sophisticated matrix techniques will be used to get solutions. Other variational techniques other than the calculus of variations may be used. It is expected that the method of steepest descent, the method of least action, and the Rayleigh-Ritz variational technique may be used. However all this will generate greater accuracy in the prediction of optimum values.

CONCLUSIONS

This paper describes mathematical technique that will aid in the design of AC Helmholtz coils that is a candidate test apparatus for low frequency magnetic field susceptibility testing on military equipment and subsystems. Use of this coil would reduce electromagnetic compatibility (EMC) qualification test time, and subject the equipment under test (EUT) to a relatively uniform field. The use of Helmholtz coils would be a great advantage to magnetic field susceptibility testing, and compared to a loop antenna would greatly reduce the number of test points needed [6]. It is tedious to design Helmholtz coils with an optimum number of turns when some quantities increase and others decrease as a function of some physical parameters. An attempt to find optimum values for a Helmholtz coils by testing would lead to many trials with no guarantee of conclusive results. The variational principle can find optimum values and aid in the design of Helmholtz coils. The variational principle will become more accurate in future models when time dependent magnetic and electric fields are used in the Lagrange multiplier equations. However, with each new variable included to enhance accuracy, the equations become much more complex. Sophisticated matrix techniques will have to be used and other variational techniques such as the Rayleigh-Ritz technique may prove superior to the Lagrange multiplier technique when all the variables are included. That is the goal. The variational technique used here is limited in the number of variables but yield results that make physical sense given the limitations and incompleteness of the model. Future work on these variational principles applied to Helmholtz coils will prove greater accuracy.

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