Dynamic Analysis of Elastic Solids by MPS Method

MooSeop Song¹*, Seiichi Koshizuka¹, Yoshiaki Oka¹
¹Nuclear Engineering Research Laboratory, The University of Tokyo, Tokai, Naka, Ibaraki, 319-1188, Japan

This study provides dynamic analysis of solids by MPS (Moving Particle Semi-implicit) method. A two-dimensional code based on particle interaction models of the MPS method is developed to simulate elastic behaviors, large deformation and fracture in solids. Meshes are not needed in the MPS method, so that fracture as well as large deformation can be analyzed without mesh distortion. We propose a new particle method for dynamic simulation of elasticity, where the equation of motion in elastic solids characterized by material density, Young’s modulus and Poisson’s ratio is discretized to particle interaction between neighboring two particles. Rotation of particles is considered for the conservation of angular momentum in the discretized formulation. We obtain limit of numerical stability in an elastic beam analysis. Large deformation and fracture of a solid are simulated. The result shows good agreement with an experiment.

KEYWORDS: Structure Analysis, MPS Method, Elastics, Dynamic Analysis, Large Deformation

I. Introduction

This study is aimed at a new method to analyze large deformation and fracture in structures. A two-dimensional dynamic code based on particle interaction models of MPS (Moving Particle Semi-implicit) method is developed.

MPS was developed for incompressible fluid dynamics (Koshizuka and Oka, 1995) 1-3). Due to the mesh-less and fully Lagrangian characteristics, MPS is expected to be a powerful tool in the structure analysis that involves large deformation and fractures 4-8). Particle methods do not need any meshes, so that it is numerically simpler than mesh-based methods.

Mesh-based methods have been widely studied and developed since the 1950’s. Up to the present, the mesh-based methods are studied in various fields such as heat transfer, fluid, structure and electro-magnetic analyses. Many commercial cords are used at wide research parts and industries. However the mesh-based methods have some problems. For examples, great effort and time are required for mesh generation in complicated three-dimensional analysis. Moreover, accuracy of the calculations is deteriorated by re-meshing or mesh distortion. Then, we propose a new particle method to simulate the solid mechanics.

In this new method, the equation of motion in elastic structures characterized by material density, Young’s modulus and Poisson ratio is discretized to particle interaction between neighboring two particles. Rotation of particles is considered for the conservation of angular momentum in the discretized formulation.

II. Governing Equation and Particle Interaction Models

The governing equation of two-dimensional isotropic elastic structure is

\[
\rho \frac{\partial^2 u_{ij}}{\partial t^2} = \frac{\partial}{\partial x_j} \left[ \lambda \epsilon_{ij} \delta_{ij} + 2 \mu \epsilon_{ij} \right]
\]

where \( \rho \) is density, \( u_i \) is displacement, \( \delta_{ij} \) is Kronecker’s delta and \( \lambda \) and \( \mu \) are Lame’s elastic constants in two dimensions that are expressed by

\[
\lambda = \frac{Ev}{1-v^2}
\]

\[
\mu = \frac{E}{2(1+v)}
\]

then, Eq.(1) is transformed to Eq.(6).

\[
\rho \frac{\partial^2 u_{ij}}{\partial t^2} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}
\]

In the MPS method, a particle interacts with its neighboring particles covered with a weight function \( W(r_{ij}) \), where \( r_{ij} \) is a distance between two particles \( i \) and \( j \) in Fig.1. The weight function of MPS method is as follows

\[
W (r_{ij}) = \begin{cases} 
\frac{r_{ij}}{r_e} - 1 & (r_{ij} \leq r_e) \\
0 & (r_{ij} > r_e)
\end{cases}
\]

* Corresponding author, Tel. +81-3-5841-2955,
E-mail: song@utnl.jp
An elastic solid is expressed by particles, and then coordinates, velocity, angle, and angular velocity are given to each particle. In order to simulate the large deformation, particle displacement is not given as degrees of freedom but it calculates from coordinates.

Strain tensor and rotation tensor are 
$$\varepsilon = \frac{1}{2} \left[ \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right]$$
$$\omega = \frac{1}{2} \left[ \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right]$$

Angular momentum of particle is 
$$I \frac{\partial^2 \theta}{\partial t^2} = M$$
where \(\theta\) is rotation angle which is angle to be twice of rotation tensor. \(I\) is moment of inertia, and \(m\) is mass of a particle.

III. Dynamic Analysis of Elastic Solid

1. Inter-particle displacement

Figure 2 shows the interactions between particle \(i\) and \(j\). When the particle \(j\) (white particle) moves to a new position of \(j\) (gray particle), position vectors are calculated, \(r_{ij}\) is the position vector of the initial state between \(i\) and \(j\) Eq. (13), \(r_{ij}\) is the rotated vector from \(r_{ij}\) Eq. (14) and \(r_{ij}\) is the rotated vector from \(r_{ij}\) Eq. (15).

The vector of displacement by strain is
$$\Delta u_i = \Delta u_i - \Delta u_i^o = r_j - r_i$$
and this is divided into normal and shear vectors.

$$\Delta u_n^i = \Delta u_i \cdot \frac{r_j}{r_{ij}}$$
$$\Delta u_t^i = \Delta u_i - \Delta u_n^i$$
In Eqs.(21) and (22), $\varepsilon_n$ is the normal strain and $\varepsilon_s$ is the shear strain between particle $i$ and $j$.

$$\varepsilon_n = \frac{\Delta u_n^e}{r_{ij}} \quad (21)$$

$$\varepsilon_s = \frac{\Delta u_s^e}{r_{ij}} \quad (22)$$

3. Stress

Normal and shear stresses are obtained by using normal and shear component vectors $u_n^e$ and $u_s^e$. Isotropic pressure is obtained at the particle locations but stresses are obtained between the particles.

$$\sigma_{ij} = 2\mu\varepsilon_n = 2\mu\frac{u_n^e}{r_{ij}} \quad (23)$$

$$\tau_{ij} = 2\mu\varepsilon_s = 2\mu\frac{u_s^e}{r_{ij}} \quad (24)$$

4. Isotropic Pressure

Pressure is represented by divergence of strain. The divergence operator is discretized by the divergence model of the MPS method at the location of a particle.

$$\varepsilon_{ik} = \text{div}(u) = \text{div}(u^e)$$

$$= \frac{d}{n^0} \sum_{j\neq i} (u_n^e - u_n^e) \cdot \frac{(r_j - r_i)}{r_{ij}^2} W(r_{ij}) \quad (25)$$

$$= -\lambda \varepsilon_{ik}$$

$$= -\lambda \text{div}(u^e) = -\lambda \frac{d}{n^0} \sum_{j\neq i} \frac{(\Delta u_n^e)_{ij}}{r_{ij}} W(r_{ij}) \quad (26)$$

where $d$ is the number of spatial dimension and $W(r_{ij})$ is the weight function.

5. Translation

Translation of particles is obtained from normal stress, shear stress and isotropic pressure. Translation of particles is described by Eq.(27).

$$\rho \frac{\partial \mathbf{r}}{\partial t} = \nabla \cdot \mathbf{\sigma} \quad (27)$$

If the divergence model is applied to Eq.(27)

$$\rho \frac{\partial \mathbf{r}}{\partial t} = \frac{d}{n^0} \sum_{j\neq i} \frac{\sigma_{ij} \cdot (r_j - r_i)}{r_{ij}^2} W(r_{ij}) \quad (28)$$

The term $\sigma_{ij} \cdot (r_j - r_i)$ of Eq.(28) is the force acting on perpendicular face to vector $(r_j - r_i)$. If this is denoted by $F_{ij}$, normal and shear components are $\sigma_{ij}Eq.(23)$ and $\tau_{ij}Eq.(24)$, respectively. Eqs. (23) and (24) are rewritten to Eqs. (29) and (30).

$$\rho \left[ \frac{\partial r_{ij}}{\partial t} \right]_1 = \frac{d}{n^0} \sum_{j\neq i} \frac{\sigma_{ij}}{r_{ij}} W(r_{ij}) \quad (29)$$

$$\rho \left[ \frac{\partial r_{ij}}{\partial t} \right]_2 = \frac{d}{n^0} \sum_{j\neq i} \frac{\tau_{ij}}{r_{ij}} W(r_{ij}) \quad (30)$$

The direction of the normal stress is $(r_j - r_i)$. The direction of shear stress is $90^\circ$ of $r_j - r_i$ to the anticlockwise rotation. Isotropic pressure is calculated similar to the normal stress Eq.(31).

$$\rho \left[ \frac{\partial r_{ij}}{\partial t} \right]_3 = -\frac{d}{n^0} \sum_{j\neq i} \frac{p_{ij}}{r_{ij}} W(r_{ij}) \quad (31)$$

where, $p_{ij}$ is to be

$$p_{ij} = \frac{p_i + p_j}{2} \quad (32)$$

the velocity and coordinates of particles are updated.
\[ v_{i}^{n+1} = v_{i}^{n} + \Delta t \left( \frac{\partial r_{i}}{\partial t} + \frac{\partial r_{i}}{\partial t} + \frac{\partial r_{i}}{\partial t} \right) \]  \hspace{1cm} (33)

\[ r_{j}^{n+1} = r_{j}^{n} + \Delta t v_{j}^{n+1} \]  \hspace{1cm} (34)

6. Rotation

Figure 3 shows the schematic representation of moment \( M_{ij} \) generated by shear stress Eq.(35).

\[ M_{ij} = \Delta \overrightarrow{F}_{ij} \times (\overrightarrow{r}_{j} - \overrightarrow{r}_{i}) \]  \hspace{1cm} (35)

where, \( \Delta \overrightarrow{F}_{ij} \) is a force applied to one particle Eq.(36).

\[ \Delta \overrightarrow{F}_{ij} = \frac{d n}{d r_{ij}} \tau_{ij} W(\overrightarrow{r}_{ij}) \cdot \hat{c} \]  \hspace{1cm} (36)

where \( \hat{c} \) is the 90° rotated unit vector of the inter-particle position vector Eq.(37).

\[ \hat{c} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\overrightarrow{r}_{j} - \overrightarrow{r}_{i}}{r_{ij}} \]  \hspace{1cm} (37)

The generated moment is applied to two particles by half from a viewpoint of angular momentum conservation.

\[ I \frac{\partial \omega_{i}}{\partial t} = -\frac{1}{2} \sum_{j} M_{ij} \]  \hspace{1cm} (38)

where \( \omega_{i} \) is the angular velocity, \( I \) is moment of inertia. The moment of inertia of a particle is given as the particle shape is a square Eq.(11). Rotation of particle is calculated by updating the angular velocity and the rotation angle.

\[ \omega_{i}^{n+1} = \omega_{i}^{n} + \Delta t \left( \frac{\partial \omega_{i}}{\partial t} \right)^{n} \]  \hspace{1cm} (39)

\[ \theta_{i}^{n+1} = \theta_{i}^{n} + \Delta t \theta_{i}^{n+1} \]  \hspace{1cm} (40)

IV. Numerical Stability by Time Step

As shown in Fig. 4, a two-dimensional beam with width and length of 0.01m and 0.05m is discretized by 11×51=561 particles. Distance between particles set to 1mm. The particles of the bottom are fixed in the directions of X and Y. The speed of 0.1~10.0 m/sec is given the particles of the top at first time step. The stable limit of \( \Delta t \) is investigated. Poisson’s ratio and density are fixed as 0.3 and 1000 kg/m³, respectively.

![Fig.4 Beam model for investigation of numerical stability](image)

The dependency of numerical instability on \( \Delta t \) is investigated. The limit of \( \Delta t \) is shown in Fig. 6 by changing Young’s modulus and initial velocity. In the figure, the lower left is a stable domain. If Young’s modulus increases by 10 times, \( \Delta t \) must be smaller to about by 0.31. The same result is obtained for any initial speed, if Young's modulus is larger than 1.0×10⁶Pa. In the case of \( E=1.0 \times 10^6 \)Pa, limit of \( \Delta t \) depends on the initial speed.

![Fig.5 The limit of time step \( \Delta t \) vs. Young’s modulus \( E \)](image)
The Courant number \( C \) is defined as Eq.(41).
\[
C = \frac{dt}{l_0} V
\]  
(41)

The conditions of numerical stability are expressed as Eq.(42).
\[
dt_{\text{limit}} < \frac{l_0}{V} C_{\text{limit}}
\]  
(42)

The limit of \( dt \) is influenced by spacing of particles \( l_0 \) and initial velocity of top particles \( V \). For the initial velocity of particles \( V = 5.0, 8.0 \) and \( 10.0 \) m/s, the limit of the Courant number is evaluated in the case of \( E = 1.0 \times 10^6 \) Pa in Table 1. The limit of the Courant numbers is equal to 0.145 in any cases, numerical stability is restricted by the Courant number.

When initial velocity of particle \( V \) is 0.1 and 1.0 m/s, \( dt \) is determined by another numerical stability condition at \( E = 1.0 \times 10^5 \) Pa or less, since the solid is too soft, the calculation is unstable in \( V = 8.0 \) m/s and 10.0 m/s. Let us consider the spring that is connected between the particles shown in Fig. 1.

<table>
<thead>
<tr>
<th>( V ) (m/s)</th>
<th>( \frac{d_{\text{limit}}}{l_0} ) (( \times 10^{-5} ))</th>
<th>( C_{\text{limit}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0 m/s</td>
<td>1.90</td>
<td>0.145</td>
</tr>
<tr>
<td>8.0 m/s</td>
<td>1.81</td>
<td>0.145</td>
</tr>
<tr>
<td>10.0 m/s</td>
<td>1.45</td>
<td>0.145</td>
</tr>
</tbody>
</table>

Generally, relation between the oscillating period \( T \) and the spring constant \( k \) is as follows.
\[
T = 2\pi \sqrt{\frac{m}{k}}
\]  
(43)

By the MPS method, the following spring constants are given as a result of the particle interaction model.
\[
k = \frac{2\mu}{\rho n^0} \left( \frac{2}{r_y^2} \right)^2 W(r_y)
\]  
(44)

Time step \( dt \) must be smaller than the oscillating period \( T \) of spring, which is considered as a numerical stability condition.
\[
dt < \alpha T
\]  
(45)

Since Lame’s constant \( \mu \) is proportional to Young’s modulus \( E \),
\[
dt \propto \frac{1}{\sqrt{E}}
\]  
(46)

Equation (46) means that \( dt \) must be smaller to \( \sqrt{1/10} \) times when Young’s modulus is increased 10 times.

This explains the line in Fig. 6. A Courant condition using the speed \( V_e \) of an elastic wave is considered.

\[
dt < \frac{l_0}{V_e} C
\]  
(47)

The speed of an elastic wave is proportional to Young’s modulus.
\[
V_e \propto \sqrt{\frac{E}{\rho}}
\]  
(48)

This is similar to Eqs.(43)–(46). Thus we can also explain, that numerical stability is governed by the Courant condition determined by the velocity of elastic wave. The limit of \( dt \) in the MPS method is clarified by this research.

V. Simulation

1. Large deformation and energy conservation

Dynamic collision calculation of two-dimensional elastic rubber rings, which show compression and tension simultaneously. The initial conditions are shown in Fig. 6. The initial spacing of the rings is 0.1m, distance between elastic body centers is 0.2m. The thickness is 0.025m. The two elastic bodies are discretized by 60 particles, the left elastic ring has a positive velocity of 2.0 m/sec, and the right elastic ring has a negative velocity of -2.0 m/sec. Material properties\(^9\) are Young’s modulus: \( 2.0 \times 10^6 \) Pa, Poisson’s ratio: 0.48 and density: 940 kg/m\(^3\).

![Collision model of two-dimensional elastic rubber rings](image_url)

Figure 7 is the results. At first, two elastic rings move according to the initial velocities. They collide at about 0.024 seconds. Two elastic rings exhibit compressible and tensile stress simultaneously with large deformation. The elastic rubber rings rebounded, and they are left mutually, leaving vibration inside. This is because the kinetic energy is changed to the internal energy.
Fig. 7  Collision of two rubber rings (and change of pressure and velocity)
Fig. 8 shows the change of total energy. Energy conservation is very well before and after collision though the total energy is not conserved during collision.

2. Fracture

The length and width of an elastic solid is 60mm × 11mm using the material coefficients of Young's modulus: 1.0 × 10^7 Pa, Poisson’s ratio: 0.3 and density: 6000 kg/m³. In this calculation, a new fracture algorithm is employed. The initial particle interval $l_0$ is changed to $l_0(1+\delta)$. The interaction between particles is lost, when $1+\delta$ exceeds 1.14; that is the value of weight function $W$ is compulsorily set to 0. This is the same to the fracture model of the strain to be 0.14.

The results of experiment and calculation are shown in Fig. 9. Although Young's modulus of the solid looks larger in the experiment, the qualitative behaviors are similar. No matter what falling angle the solid has, it is divided into three fragments in the experiment. The simulation result successfully reproduces this behavior.
VI. Conclusion

The following conclusions are obtained in this study.
- A two-dimensional dynamic elastic analysis technique using the MPS method is developed.
- The numerical stability condition in dynamic elastic analysis is explained as Courant conditions.
- Large deformation and fracture are calculated using this technique. This technique is to be extended to three-dimensional analysis in future.

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References