Higher Order Sampling and Recovering of Lowpass Signals

Gessê Eduardo Calvo Nogueira and Ademar Ferreira

Abstract—A higher order sampling function applied to lowpass signals is described. To recover the original signal from its periodic nonuniform samples, a formulation of the sampling function is introduced, based on the occurrence of local gaps in the sampled signal spectrum. As a result, the interpolant is greatly simplified.

Index Terms—High order sampling, nonuniform interpolators, nonuniform sampling.

I. INTRODUCTION

The $N$th-order sampling of lowpass signals is used in several areas, such as in medical pulsed Doppler ultrasonic velocimeters [1] and in multirate systems [2]. In the literature, known methods$^1$ for signal reconstruction from its periodic nonuniform samples are often accomplished by $N$ interpolators (see, e.g., [3]). The implementation of such interpolators, which are often approximated by FIR filters, requires various considerations. Vaughan et al. [4], for example, showed that the length and the dynamic range (number of bits) needed for digital implementation of the interpolators proposed by Kohlenberg [5] (second-order sampling) for the lowpass case, are progressively larger as the Kohlenberg’s factor $k$ departs from its optimum value: $k = T/2$ (uniform sampling), $T$ is the sampling interval of each uniform sequence, where the interpolator becomes a lowpass filter. Although, to our knowledge, a comparative study on the performance of such interpolators, concerning filter length, computational efficiency, etc., is not found in the literature, one can verify that the known periodic nonuniform interpolation methods require considerably more complicated processing than the one for uniform sampling.

This correspondence describes in Section II a new higher order sampling function and restrictions to this sampling pattern intended to simplify the signal reconstruction. In Section III, restrictions are derived to recover second-order sampled lowpass signals at minimum average sampling rate by using only lowpass filters and a frequency shifter. Third and fourth-order sampling cases are also discussed in Section IV, where the signal reconstruction is accomplished using only lowpass filters but with average sampling rate larger than the minimum.

II. HIGHER ORDER SAMPLING

Given a real and lowpass signal $s(t)$ bandlimited to the frequency interval $|f| < B$ Hz, i.e., its highest spectral component is located at $B' < B$, we define the $N$th-order sampling of this signal as

$$s_N(t) = s(t) \cdot \sum_{i=1}^{N} e_i \sum_{n=-\infty}^{\infty} \delta(t - nT - k_i)$$

where $i = 1, 2, \ldots, N$, $n = \pm 1$, $\pm 2$, $\cdots$, $e_i$ are amplitudes (nonzero real values), $T$ is the sampling period of each uniform sampling sequence, each one presenting a delay $k_i$, and $\delta(t)$ is the Dirac delta function.

The average sampling frequency in (1) is $T_N = N/T$. Here, we refer to $f_N$ as minimum when it is equal to the Nyquist rate, i.e., when $T_N = 2B$. To simplify the following analysis, we impose $k_i = 0$ and $k_i < T$ for $i \neq 1$. The following notation is used: $S(f)$ designates the Fourier transform of $s(t)$; since $s(t)$ is real and lowpass, its spectrum can be represented by $S(f) = S^+(f) + S^-(f)$, where $S^+(f)$ and $S^-(f)$ occupy, respectively, positive and negative frequencies, from here on referred to as upper and lower bands, correspondingly, of $S(f)$ or any of its replicas. The Fourier transform of (1) is

$$S_N(f) = \sum_{i=1}^{N} e_i \sum_{n=-\infty}^{\infty} S(f - n/T) \exp(-j2\pi n k_i/T).$$

The replicas of $S(f)$ in (2), in general, are $1/T$ spaced apart. For $N = 1$ (uniform sampling case), obviously, the recovery of $s(t)$ is possible if $B \leq 1/2T$. For $N > 1$, if the spacing between consecutive replicas, even locally, can be made larger than $1/T$, then the band of $s(t)$ can be increased. We discuss now some conditions to obtain this enlargement by generating gaps between consecutive replicas.

Notice in (2) that at every frequency $n/T$, the phases and magnitudes of the $N$ replicas can be such that the resulting sum is constructive or destructive, depending on the amplitudes $e_i$ and delays $k_i$. When destructive, that is, when

$$\sum_{i=1}^{N} e_i S(f - n/T) \exp(-j2\pi n k_i/T) = 0$$

we say that a gap occurs in this location of the spectrum. In this case, the empty spectral spacing between the two remaining replicas adjacent to this gap, situated at $n - 1$ and $n + 1$, is magnified to $2/T$. For two consecutive gaps, the empty spacing is $3/T$, and so forth. Once one gap is produced, to recover the original signal, the selection of bands of the replicas adjacent to the generated gap is affected by spectral windowing, as will be shown in the next section.

III. RECOVERY OF SECOND-ORDER SAMPLED SIGNALS

To originate one gap at a specific frequency $n/T$, one can impose the condition (3) to this position. Instead of this, in the case $N = 2$, it is easy to search for all possible gaps at all positions $n$ in (2), as we show next.

For second-order sampling, the spectrum of (2) is

$$S_2(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \left[ e_1 + e_2 \exp(-j2\pi n k_2/T) \right] S(f - n/T).$$

To find $e_1$, $e_2$, and $k_2$ for all possible $n$ positions in (4) that satisfy (3), we impose in (4)

$$e_1 + e_2 \exp(-j2\pi n k_2/T) = 0.$$
will be further discussed. On both the above cases, because the \( e_i \),s are real values, the gaps are placed symmetrically with respect to the 
\( n = 0 \) axis of the spectrum (4). To simplify the recovery of \( s(t) \), the 
pREFERRED case is the one that produces gaps at \( n = \pm 1 \) because the 
needed interpolator is a single lowpass filter. In both cases, regarding 
\( e_i \), however, one can verify that there is a unique choice of \( k_2/T \) to 
produce gaps at \( n = \pm 1 \). That is, the gaps at \( n = \pm 1 \) (as well as at 
\( n = \pm 3, \pm 5, \ldots \) because they are periodic) take place only when 
\( e_1 = e_2 \) and \( k_2/T = 1/2 \), which is a case of uniform sampling. When 
\( e_1 = -e_2 \), there is a gap at the origin for any delay, as mentioned. 
Excepting this gap at \( n = 0 \), the closest gaps to the origin can only occur 
at \( n = \pm 2 \) (and at \( n = \pm 4, \pm 6, \ldots \) when \( k_2/T = 1/2 \), which is also 
uniform sampling. For other positions \( n \), there is some freedom 
in the choice of the delay. In the case \( e_1 = -e_2 \), for example, gaps 
take place at \( n = \pm 3 \) (as well as at \( n = 0, \pm 6, \pm 9, \ldots \) by choosing 
k_2/T = 1/3 or \( k_2/T = 2/3 \). These cases are not explored here. 

Since only when \( k_2/T = 1/2 \) and \( e_1 \) and \( e_2 \) is it possible to recover 
\( s(t) \) using a single lowpass filter, we discuss now the case \( e_1 = -e_2 \), 
where a gap occurs at \( n = 0 \) for any delay \( k_2 \). To simplify the following 
analysis, let \( e_1 = 1 \) and \( e_2 = -1 \). Under this condition, consider the 
spectrum \( S_2(f) \) of (4) multiplied by an ideal spectral window \( G(f) = T \) 
with passband \[ |f| \leq 1/T \]. This filter selects the lower bands of the 
replicas at \( n = \pm 1 \). The resultant spectrum after filtering is 

\[
F(f) = S^{-1}(f - 1/T)[1 - \exp(-j\alpha)] + S^{-1}(f + 1/T)[1 - \exp(j\alpha)] 
\]  

where \( \alpha = 2\pi k_2/T \). Here, we can verify that if \( s(t) \) is bandlimited 
to \( B' < 1/T \), the lower bands of the replicas at \( n = 1 \) and \( n = -1 \) can occupy the frequency interval \([-1/T, 1/T]\) without spectral 
overlapping the replica at \( n = 0 \) (which was suppressed). The inverse 
Fourier transform of (6), written in a more convenient form, is 

\[
f(t) = 2C \left[ s(t) \cos \left( 2\pi \frac{1}{T} t - \beta \right) + \hat{s}(t) \sin \left( 2\pi \frac{1}{T} t - \beta \right) \right]
\]  

(7) 

where \( \hat{s}(t) \) is the Hilbert transform of \( s(t) \), \( C = 2 \cdot \sin(\alpha/2) \), and 
\( \beta = \arctan(\sin(\alpha)/[1 - \cos(\alpha)]) \). To recover \( s(t) \), a frequency shifting 
followed by lowpass filtering is necessary, as follows: 

\[
s(t) = \left[ f(t) \cdot \frac{1}{2C} \cos \left( 2\pi \frac{1}{T} t - \beta \right) \right] * g(t)
\]  

(8) 

where \( g(t) \) is the inverse Fourier transform of a lowpass filter with passband 
\[ |f| \leq B' \]. In this case, the average sampling rate is minimum 
because \( \frac{T}{f_s} = 2/T \), and \( B \leq 1/T \). Notice that although this recovering 
method uses only two lowpass filters, a modulation process is 
necessary. Thus, this method can be easily implemented only in systems 
where an oscillator is available, as in weather radar and Doppler 
ultrasound systems. For Doppler velocimetry, however, \( F(f) \) contains 
all relevant information about \( s(t) \), which in this case is the Doppler 
shifted signal.2 That is, for these applications, the most relevant data are 
the mean velocities and the velocity distribution widths. These data can be 
digitally computed (estimated) directly from the nonuniformly sampled 
signal spectrum (6), i.e., neither signal reconstruction nor delay 
knowledge are necessary [1]. 

IV. RECOVERY OF \( N \)TH-ORDER SAMPLED SIGNALS

For \( N \)th-order sampling the average sampling frequency is equal 
to the Nyquist frequency \( N/T = 2B \). Now we consider \( N \geq 2 \), 
\( B > 1/T \) and \( N/T = 2B \). Only the recovering of \( s(t) \) by selecting 
the replica at \( n = 0 \) is discussed, because the filter needed is a low-pass 

\[
\left[ \begin{array}{ccc}
1 & W_2^{-1} & W_3^{-1} \\
1 & 1 & 1 \\
1 & W_2 & W_3 \\
1 & W_2^2 & W_3^2 \\
1 & W_2^3 & W_3^3
\end{array} \right] \left[ \begin{array}{c}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5
\end{array} \right] = \left[ \begin{array}{c}
0 \\
1 \\
0 \\
0 \\
0
\end{array} \right]
\]  

(9) 

\[
\left[ \begin{array}{ccc}
1 & W_2^{-1} & W_3^{-1} \\
1 & 1 & 1 \\
1 & W_2 & W_3 \\
1 & W_2^2 & W_3^2 \\
1 & W_2^3 & W_3^3
\end{array} \right] \left[ \begin{array}{c}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5
\end{array} \right] = \left[ \begin{array}{c}
0 \\
1 \\
0 \\
0 \\
0
\end{array} \right]
\]  

(10) 

Similarly, as in previous cases, there are always real values for \( e_i \) if 
k_2/T = 1 - k_2/T for k_2/T, k_3/T \neq 1/2 (distinct values) and 
k_4/T = 1/2 (other delays were not searched for). Directly computing 

2The quadrature detected Doppler signals, after being range-gated, are often 
at a lowpass position.

3The procedure is to search for delays to allow
(10), for example, with $k_2/T = 1/8$, $k_3/T = 1/2$ and $k_4/T = 7/8$, we have $e_1 = -0.3355$, $e_2 = 0.5$, $e_3 = 0.3356$, and $e_4 = 0.5$. The signal bandwidth, however, must be $B' < 3/2T$. In this case, a low-pass filter having passband $B'$ reconstructs the original signal. Since for $N = 4$ the average sampling rate is $4/T$, it is then 4/3 times higher than the minimum, and the main advantage in this case is also the simplified filter.

V. CONCLUSIONS

A higher order sampling pattern is suggested to allow recovering of a lowpass signal $s(t)$ from its nonuniform samples using only low-pass filters. For second-order sampling, it is possible to recover, by using a lowpass filter, a signal $f(t)$ from its nonuniform samples obtained with minimum rate, such that $f(t)$ contains all relevant information about the original signal for spectral estimations, which is useful for Doppler velocimetry. It is also possible to recover the original signal from $f(t)$ by a frequency shifter and another lowpass filter.

For the higher order sampling cases $n = 3$ and $N = 4$, we showed that by applying appropriate sampling restrictions, the reconstruction of the original signal is possible by using only lowpass filters but at a higher sampling rate ($3/2$ and $4/3$ times the minimum, respectively).

APPENDIX

For $N = 3$, two gaps are needed at $n = 1$ and $n = 2$ to recover $s(t)$ sampled with minimum average sampling rate, as already shown in Section IV. Entering these conditions in (3) and writing it in a more convenient way, we have

$$
\begin{bmatrix}
1 & W_2 & W_3 & e_1 \\
1 & W_2 & W_3 & e_2 \\
1 & W_2 & W_3 & e_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
$$

(A.1)

where $W_n = e^{\beta j \alpha_n}$, and $\alpha_n = 2\pi k_n/T$ or $W^* \cdot e_i = 0$, where $W$ is the $2 \times 3$ matrix in (A.1), and $e_i$ is the amplitude vector. Let the matrix $W$ be written as $W = \text{Re}(W) + j\text{Im}(W)$. For $e_1$, $e_2$ and $e_3$ to be real, it is necessary that $\text{Im}(W) \cdot e_i = 0$, or

$$
\begin{bmatrix}
-\sin(\alpha) & -\sin(\alpha_3) & e_1 \\
-\sin(2\alpha) & -\sin(2\alpha_3) & e_2 \\
-\sin(3\alpha) & -\sin(3\alpha_3) & e_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
$$

(A.2)

For distinct $\alpha \neq \pi$, for the determinant of the square matrix in (A.2) to be zero, it is necessary that

$$
\sin(2\alpha) = -\sin(2\alpha_3)
$$

(A.3)

Remembering that $0 < k_i < T$ or $0 < \alpha_i < 2\pi$, then (A.3) is always true when $\alpha_3 = 2\pi - \alpha_2$ for distinct $\alpha_i$, hence excluding $\alpha_2$, $\alpha_3 = \pi$. Solving (A.2) for these values, we find $e_2 = e_3 = e$. Taking these values into $\text{Re}(W)$, we have

$$
\begin{bmatrix}
1 & 2\cos(\alpha) & e_1 \\
1 & 2\cos(2\alpha) & e_2 \\
1 & 2\cos(3\alpha) & e_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
$$

(A.4)

The determinant of the square matrix in (A.4) is zero if $\cos(\alpha_2) = -\cos(\alpha_3)$, and the unique possible angle is $\alpha_2 = \pi/3$ (for $\alpha_2 < \alpha_3$). Solving (A.4) for this $\alpha_2$, we get $e_1 = e$. Therefore, gaps situated at $n = \pm 1$ and $n = \pm 2$ only occur if $k_2/T = 1/3$, $k_3/T = 2/3$, and $e_1 = e_2 = e_3$, which is a uniform sampling case.

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REFERENCES


Corrections to “Zolotarev Polynomials and Optimal FIR Filters”

Miroslav Vlček and Rolf Unbehauen

I. INTRODUCTION

In the above paper,1 the following corrections need to be made.

- Equation (21) should read

$$
w_m = w_0 + 2 \frac{\sin(\alpha_0)\alpha_0}{\sin(\alpha_0)} Z(u_0).
$$

- The equations on p. 722 should read

$$
b(3) \Delta = (w_x - w_p) \left(1 + w_x w_p + (1 - w_x^2) - (1 - w_p^2)\right)
$$

$$
b(2) \Delta = -w_x w_p (w_x^2 - w_p^2) - w_p^2 (1 - w_x^2) + w_x (1 - w_p^2)
$$

$$
b(1) \Delta = w_p^3 - w_x^3 + w_x^2 w_p (w_x - w_p) - (1 - w_x^2) + (1 - w_p^2)
$$

$$
b(0) \Delta = w_x w_p (w_x^3 - w_p^3) + w_p (1 - w_x^2) - w_x (1 - w_p^2).
$$

- In Table IV and V, instead of

$$
w_m = w_0 + 2 \frac{\sin(\alpha_0)\alpha_0}{\sin(\alpha_0)} Z(u_0)
$$

there should be

$$
w_m = w_0 + 2 \frac{\sin(\alpha_0)\alpha_0}{\sin(\alpha_0)} Z(u_0).
$$

- In the design procedure, the equation under item 5 on p. 727 should read

$$
w_m = \cos \omega_m T = w_0 + 2 \frac{\sin(\alpha_0)\alpha_0}{\sin(\alpha_0)} Z(u_0).$$

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